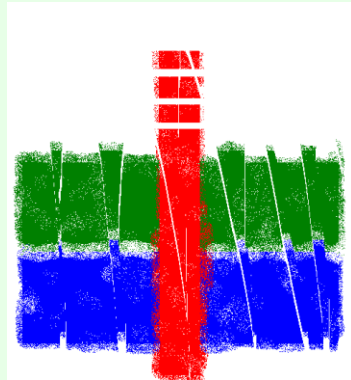
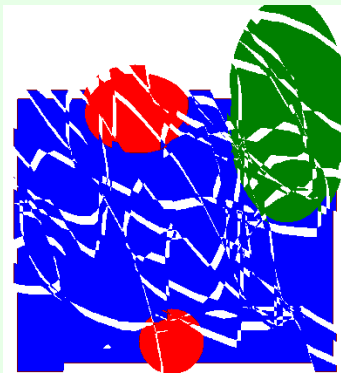


Linear programming,
integer linear programming,
mixed integer linear programming

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Example linear program

- We make reproductions of two paintings



maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

- Painting 1 sells for \$30, painting 2 sells for \$20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

Solving the linear program graphically

maximize $3x + 2y$

subject to

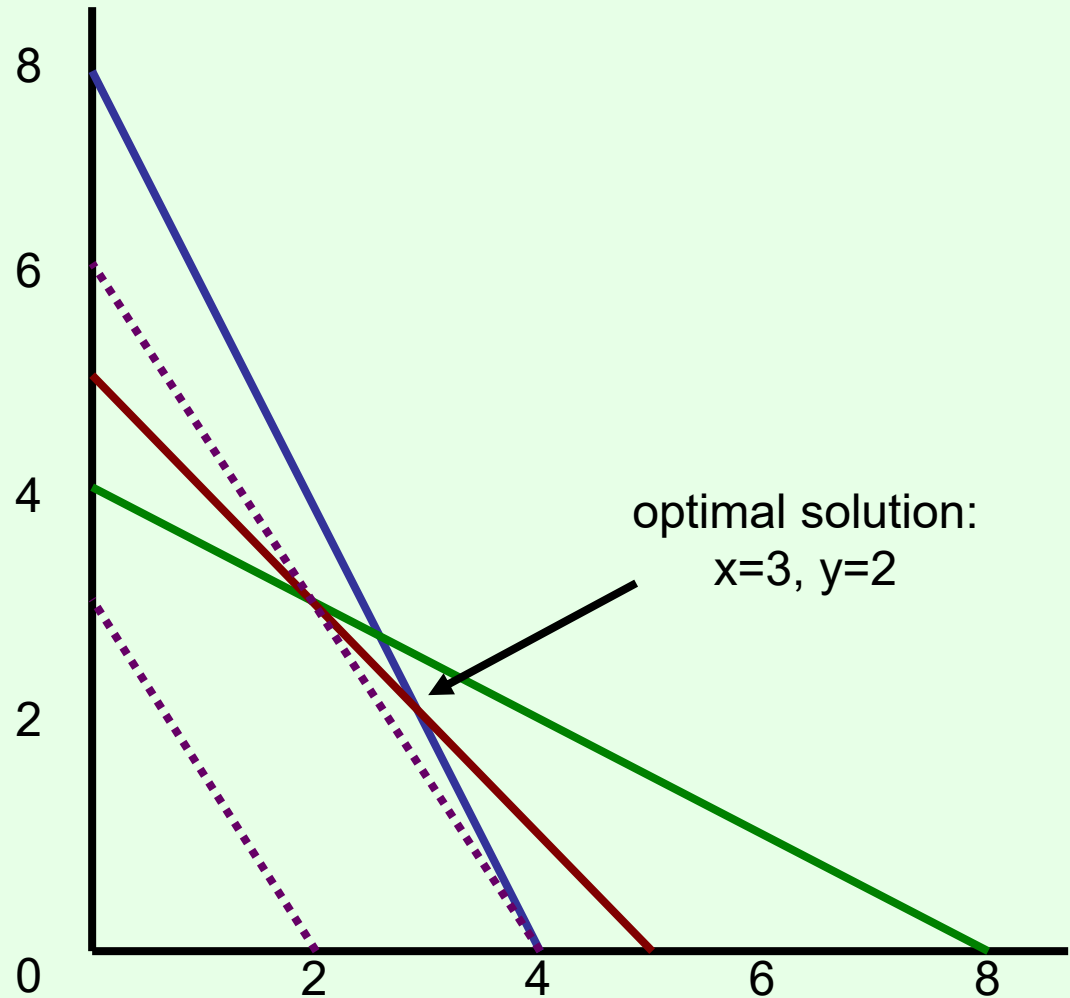
$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$



Proving optimality

maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Recall: optimal solution:

$$x=3, y=2$$

$$\text{Solution value} = 9+4 = 13$$

How do we **prove** this is optimal (without the picture)?

Proving optimality...

maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

We can rewrite the blue constraint as

$$2x + y \leq 8$$

If we add the red constraint

$$x + y \leq 5$$

we get

$$3x + 2y \leq 13$$

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)

Linear combinations of constraints

maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

$$b(4x + 2y \leq 16) +$$

$$g(x + 2y \leq 8) +$$

$$r(x + y \leq 5) =$$

$$(4b + g + r)x +$$

$$(2b + 2g + r)y \leq$$

$$16b + 8g + 5r$$

$4b + g + r$ must

be at least 3

$2b + 2g + r$ must

be at least 2

Given this,

minimize $16b +$

$$8g + 5r$$

Using LP for getting the best upper bound on an LP

maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

minimize $16b + 8g + 5r$

subject to

$$4b + g + r \geq 3$$

$$2b + 2g + r \geq 2$$

$$b \geq 0$$

$$g \geq 0$$

$$r \geq 0$$

the **dual** of the original program

- Duality theorem: any linear program has the same optimal value as its dual!

Modified LP

maximize $3x + 2y$

subject to

$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Optimal solution: $x = 2.5$,
 $y = 2.5$

Solution value = $7.5 + 5 =$
 12.5

Half paintings?

Integer (linear) program

maximize $3x + 2y$

subject to

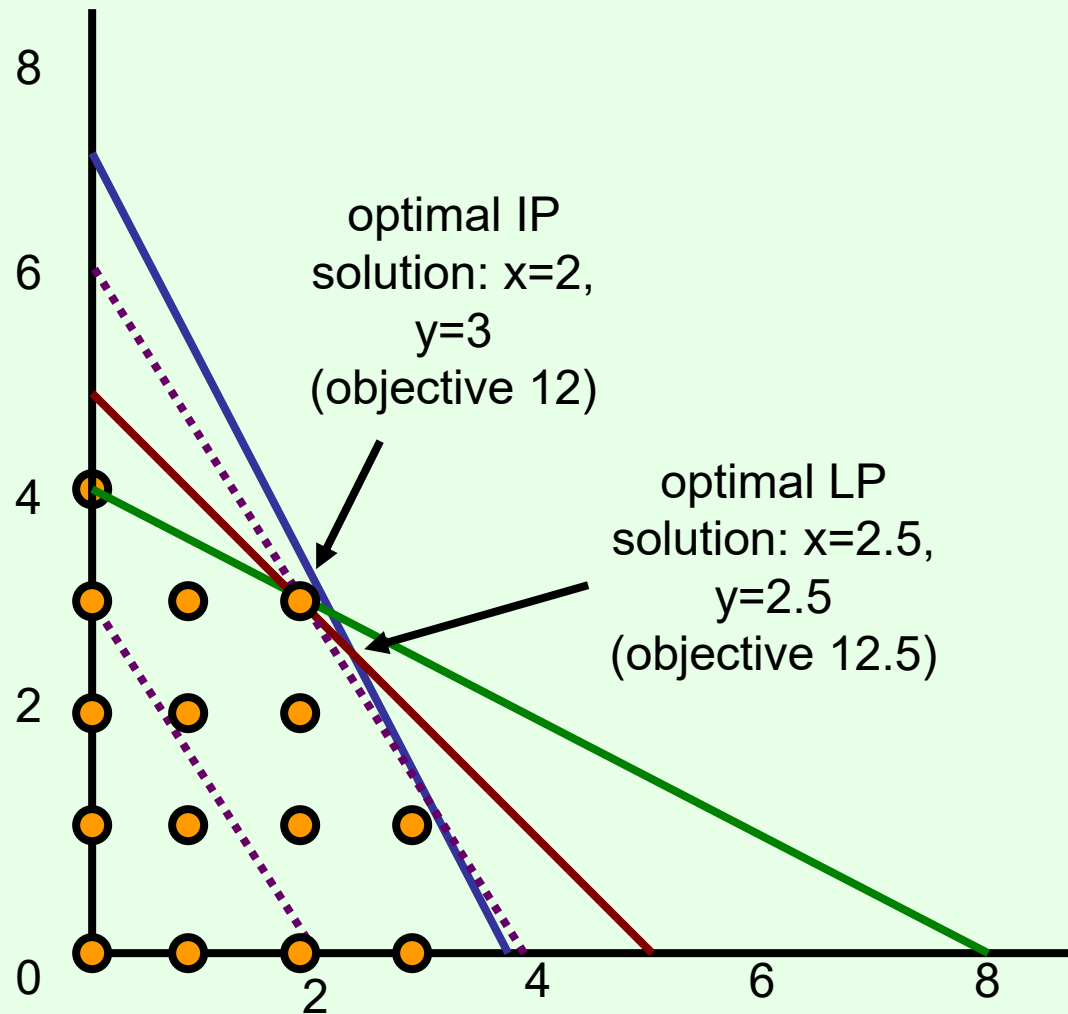
$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$x \geq 0$, integer

$y \geq 0$, integer



Mixed integer (linear) program

maximize $3x + 2y$

subject to

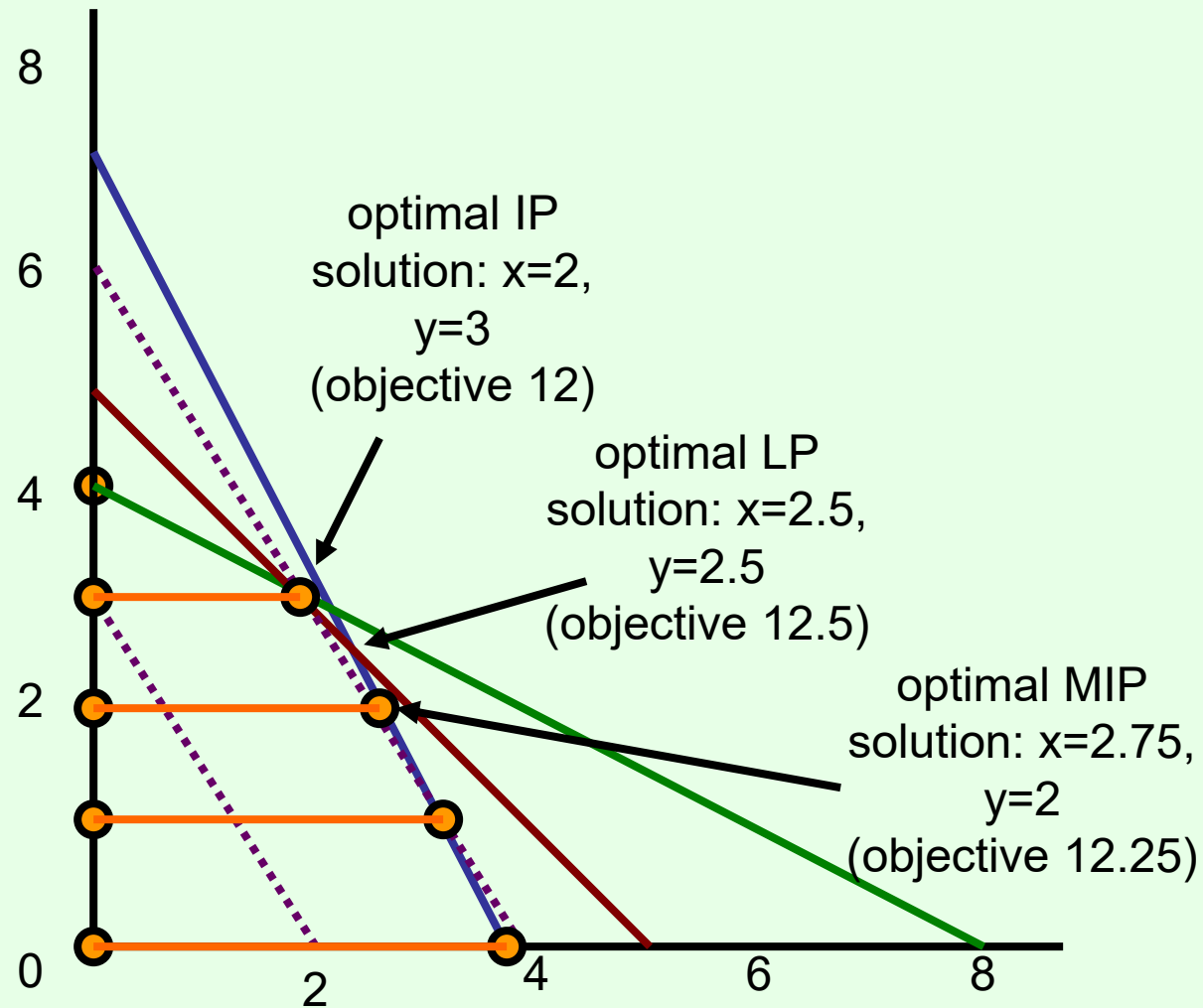
$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0, \text{ integer}$$



Solving linear/integer programs

- Linear programs can be solved **efficiently**
 - Simplex, ellipsoid, interior point methods...
- (Mixed) integer programs are **NP-hard to solve**
 - Quite easy to model many standard NP-complete problems as integer programs (try it!)
 - Search type algorithms such as branch and bound
- Standard packages for solving these
 - GNU Linear Programming Kit, CPLEX, Gurobi, ...
- **LP relaxation** of (M)IP: remove integrality constraints
 - Gives upper bound on MIP (~**admissible heuristic**)

Exercise in modeling: knapsack-type problem

- We arrive in a room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for \$11
 - There are 3 units available
- Unit of object B: 4kg, 4 liters, sells for \$4
 - There are 4 units available
- Unit of object C: 6kg, 3 liters, sells for \$9
 - Only 1 unit available
- What should we take?

The *general* version of this knapsack IP

maximize $\sum_j p_j x_j$

subject to

$$\sum_j w_j x_j \leq W$$

$$\sum_j v_j x_j \leq V$$

$$(\text{for all } j) x_j \leq a_j$$

$$(\text{for all } j) x_j \geq 0, x_j \text{ integer}$$

Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- ... but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)

