# Repeated games

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# Repeated games

- In a (typical) repeated game,
  - players play a normal-form game (aka. the stage game),
  - then they see what happened (and get the utilities),
  - then they play again,
  - etc.
- Can be repeated finitely or infinitely many times
- Really, an extensive form game
  - Would like to find subgame-perfect equilibria
- One subgame-perfect equilibrium: keep repeating some Nash equilibrium of the stage game
- But are there other equilibria?

# Finitely repeated Prisoner's Dilemma

- Two players play the Prisoner's Dilemma k times

| | cooperate | defect |
|---|---|---|
| cooperate | 2, 2 | 0, 3 |
| defect | 3, 0 | 1, 1 |

- In the last round, it is dominant to defect
- Hence, in the second-to-last round, there is no way to influence what will happen
- So, it is optimal to defect in this round as well
- Etc.
- So the only equilibrium is to always defect

# Modified Prisoner's Dilemma

- Suppose the following game is played twice

| | cooperate | $defect_1$ | $defect_2$ |
|---|---|---|---|
| cooperate | 5, 5 | 0, 6 | 0, 6 |
| $defect_1$ | 6, 0 | 4, 4 | 1, 1 |
| $defect_2$ | 6, 0 | 1, 1 | 2, 2 |

- Consider the following strategy:
  - In the first round, cooperate;
  - In the second round, if someone defected in the first round, play $defect_2$; otherwise, play $defect_1$
- If both players play this, is that a subgame perfect equilibrium?

# Another modified Prisoner's Dilemma

- Suppose the following game is played twice

| | cooperate | defect | crazy |
|---|---|---|---|
| cooperate | 5, 5 | 0, 6 | 1, 0 |
| defect | 6, 0 | 4, 4 | 1, 0 |
| crazy | 0, 1 | 0, 1 | 0, 0 |

- What are the subgame perfect equilibria?
- Consider the following strategy:
  - In the first round, cooperate;
  - In the second round, if someone played defect or crazy in the first round, play crazy; otherwise, play defect
- Is this a Nash equilibrium (not subgame perfect)?

# Infinitely repeated games

- First problem: are we just going to add up the utilities over infinitely many rounds?
  - Everyone gets infinity!
- (Limit of) average payoff: $\lim_{n \to \infty} \Sigma_{1 \leq t \leq n} u(t)/n$
  - Limit may not exist…
- Discounted payoff: $\Sigma_t \delta^t u(t)$ for some $\delta < 1$

# Infinitely repeated Prisoner's Dilemma

| | cooperate | defect |
|---|---|---|
| cooperate | 2, 2 | 0, 3 |
| defect | 3, 0 | 1, 1 |

- Tit-for-tat strategy:
  - Cooperate the first round,
  - In every later round, do the same thing as the other player did in the previous round
- Is both players playing this a Nash/subgame-perfect equilibrium? Does it depend on $\delta$?
- Trigger strategy:
  - Cooperate as long as everyone cooperates
  - Once a player defects, defect forever
- Is both players playing this a subgame-perfect equilibrium?
- What about one player playing tit-for-tat and the other playing trigger?

# Folk theorem(s)

- Can we somehow characterize the equilibria of infinitely repeated games?
  - Subgame perfect or not?
  - Averaged utilities or discounted?
- Easiest case: averaged utilities, no subgame perfection
- We will characterize what (averaged) utilities ($u_1$, $u_2$, …, $u_n$) the agents can get in equilibrium
- The utilities must be feasible: there must be outcomes of the game such that the agents, on average, get these utilities
- They must also be enforceable: deviation should lead to punishment that outweighs the benefits of deviation
- Folk theorem: a utility vector can be realized by some Nash equilibrium if and only if it is both feasible and enforceable

# Feasibility

| | |
|---|---|
| 2, 2 | 0, 3 |
| 3, 0 | 1, 1 |

- The utility vector (2, 2) is feasible because it is one of the outcomes of the game
- The utility vector (1, 2.5) is also feasible, because the agents could alternate between (2, 2) and (0, 3)
- What about (.5, 2.75)?
- What about (3, 0.1)?
- In general, convex combinations of the outcomes of the game are feasible
  - $p_1a_1 + p_2a_2 + \ldots + p_na_n$ is a convex combination of the $a_i$ if the $p_i$ sum to 1 and are nonnegative

# Enforceability

| | |
|---|---|
| 2, 2 | 0, 3 |
| 3, 0 | 1, 1 |

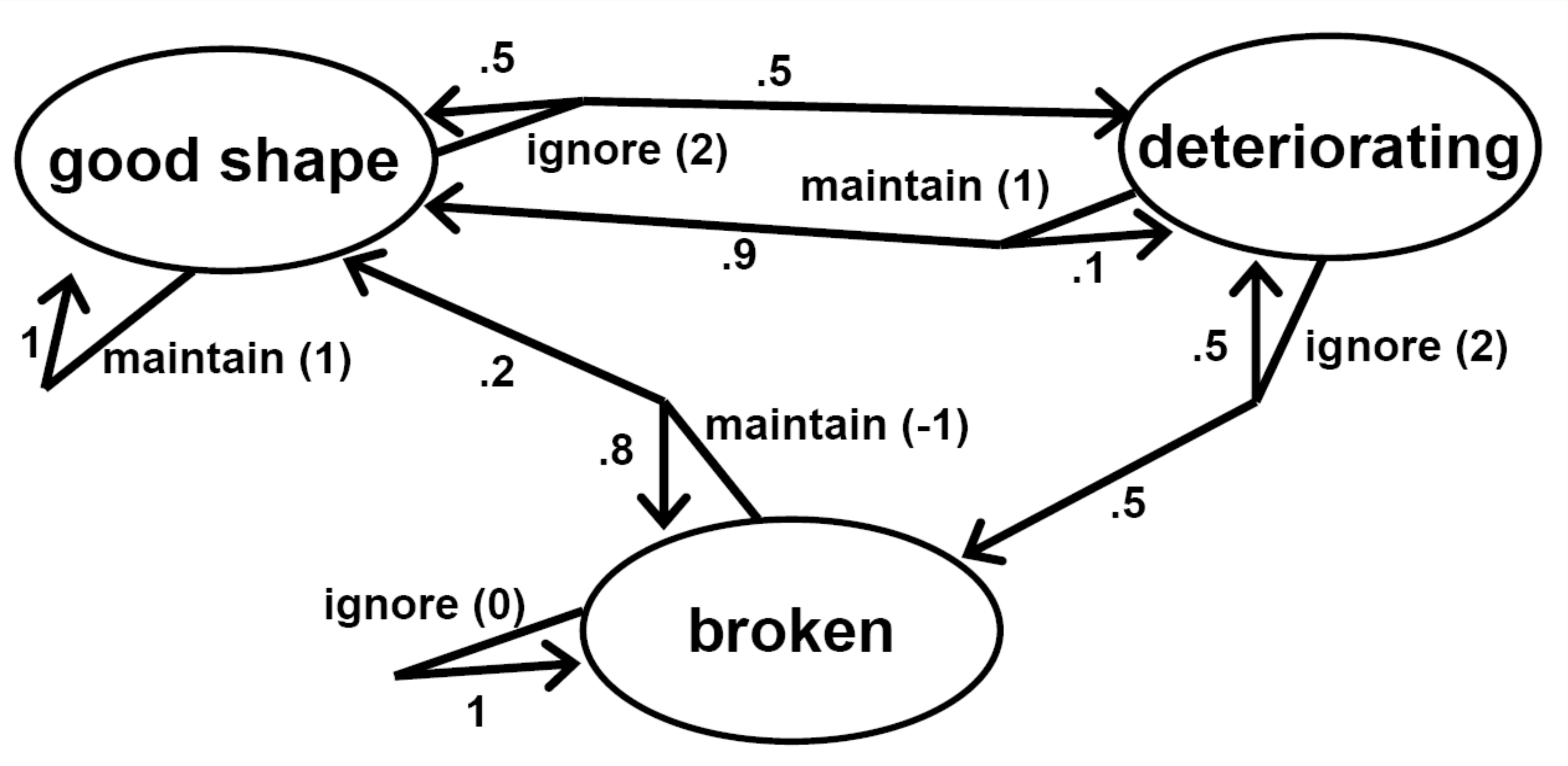
- A utility for an agent is not enforceable if the agent can guarantee herself a higher utility
- E.g. a utility of .5 for player 1 is not enforceable, because she can guarantee herself a utility of 1 by defecting
- A utility of 1.2 for player 1 is enforceable, because player 2 can guarantee player 1 a utility of at most 1 by defecting
- What is the relationship to minimax strategies & values?

# Computing a Nash equilibrium in a 2-player repeated game using folk theorem

- Average payoff, no subgame perfection
- Can be done in polynomial time:
  - Compute minimum enforceable utility for each agent
    - I.e., compute maxmin values & strategies
  - Find a feasible point where both players receive at least this utility
    - E.g., both players playing their maxmin strategies
  - Players play feasible point (by rotating through the outcomes), unless the other deviates, in which case they punish the other player by playing minmax strategy forever
    - Minmax strategy easy to compute
- A more complicated (and earlier) algorithm by Littman & Stone [04] computes a “nicer” and subgame-perfect equilibrium

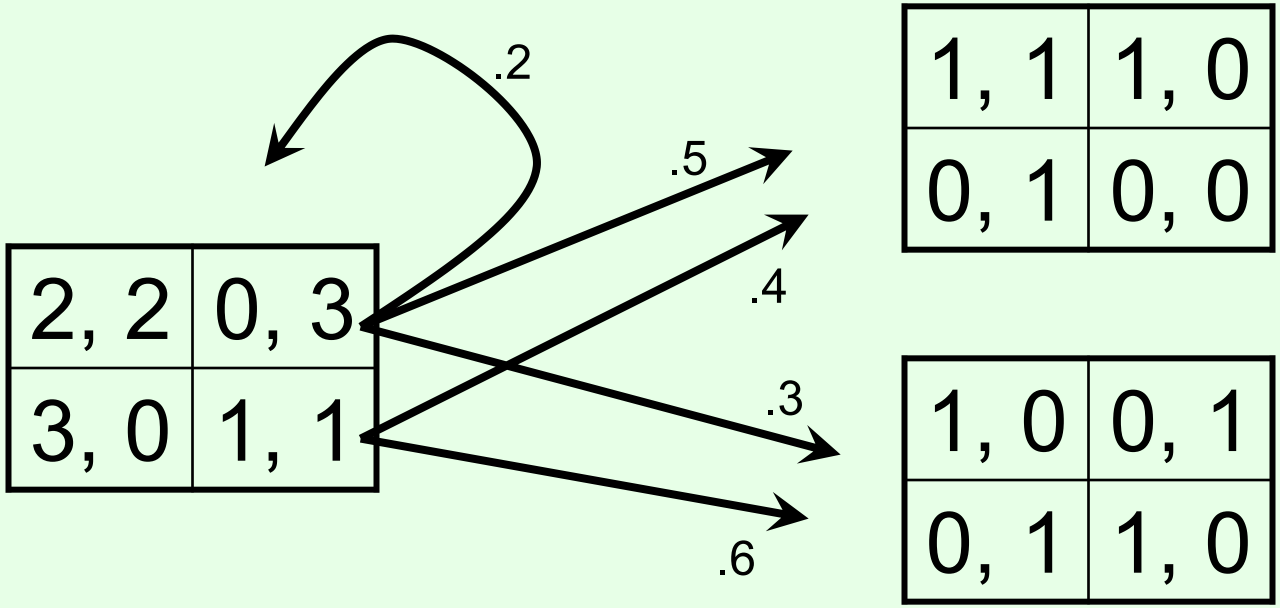
# Example Markov Decision Process (MDP)

- Machine can be in one of three states: good, deteriorating, broken
- Can take two actions: maintain, ignore



# Stochastic games

- A stochastic game has multiple states that it can be in
- Each state corresponds to a normal-form game
- After a round, the game randomly transitions to another state
- Transition probabilities depend on state and actions taken
- Typically utilities are discounted over time



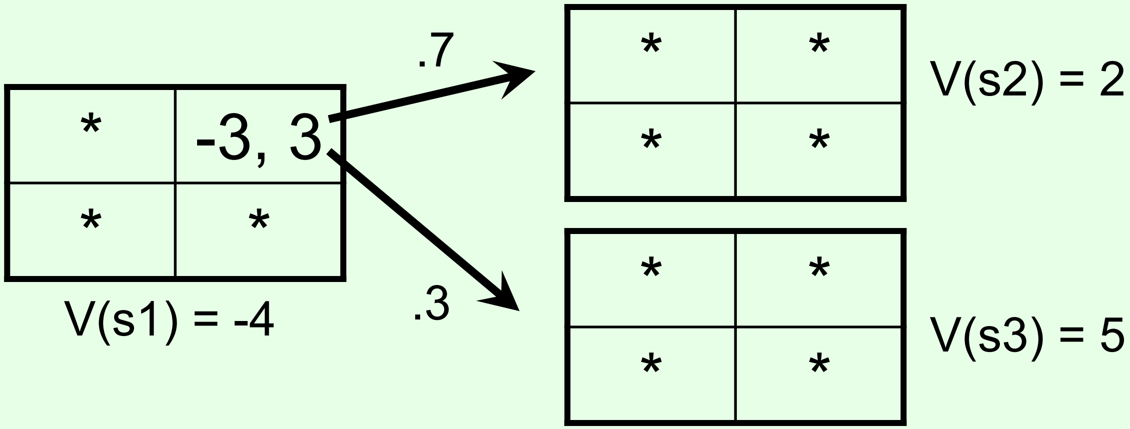

- 1-state stochastic game = (infinitely) repeated game
- 1-agent stochastic game = Markov Decision Process (MDP)

# Stationary strategies

- A stationary strategy specifies a mixed strategy for each state
  - Strategy does not depend on history
  - E.g., in a repeated game, stationary strategy = always playing the same mixed strategy
- An equilibrium in stationary strategies always exists [Fink 64]
- Each player will have a value for being in each state

# Shapley's [1953] algorithm for 2-player zero-sum stochastic games (~value iteration)

- Each state s is arbitrarily given a value V(s)
  - Player 1's utility for being in state s
- Now, for each state, compute a normal-form game that takes these (discounted) values into account




$-3 + \delta(.7*2 + .3*5)$
$= -3 + 2.9\delta$

| | |
|---|---|
| * | $-3 + 2.9\delta$, $3 - 2.9\delta$ |
| * | * |

s1's modified game

- Solve for the value of the modified game (using LP)
- Make this the new value of s1
- Do this for all states, repeat until convergence
- Similarly, analogs of policy iteration [Pollatschek & Avi-Itzhak] and Q-Learning [Littman 94, Hu & Wellman 98] exist