Due Date: November 20, 11:59pm

Problem 1: [10pts] Given a ground set $U = \{e_1, e_2, \ldots, e_n\}$, a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ where each $S_j \subseteq U$ and $\bigcup_{j=1}^{m} S_j = U$, and a weight function $w : S_j \to \mathbb{R}^+$, we want to find a set cover $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup_{S_j \in \mathcal{C}} S_j = U$ and the total weight of sets in $\mathcal{C}$ is minimized. Below is a linear program relaxation for the problem.

$$\begin{aligned}
\min \quad & \sum_{j=1}^{m} w(S_j) \cdot x_j \\
\text{subject to} \quad & \sum_{e_i \in S_j} x_j \geq 1 \quad \text{for } 1 \leq i \leq n \\
& x_j \geq 0 \quad \text{for } 1 \leq j \leq m.
\end{aligned}$$

Let f be the maximum number of sets in $\mathcal{S}$ in which an element of U appears. Consider the algorithm for this problem that first computes an optimal fractional solution x^* then returns $\{S_j \mid x_j^* \geq 1/f\}$. Prove this algorithm returns a feasible solution and prove the best approximation guarantee for it that you can.

Problem 2: [10pts] Let G be an undirected cycle with n vertices. Write the Laplacian matrix of G. Prove that its eigenvalues and eigenvectors are

$$\lambda_k = 2 - 2\cos\left(2\pi\frac{k}{n}\right) \quad \text{and} \quad u_k = \begin{bmatrix} \cos\left(0\pi\frac{k}{n}\right) \\ \cos\left(2\pi\frac{k}{n}\right) \\ \cos\left(4\pi\frac{k}{n}\right) \\ \cdots \\ \cos\left(2(n-1)\pi\frac{k}{n}\right) \end{bmatrix}$$

respectively, for $0 \leq k \leq n-1$.

Problem 3: [10pts] Let $\mathcal{U}$ be the universe of items to be hashed, and m be the size of the hash table. A family $\mathcal{H}$ of hash functions is *uniform* if choosing a hash function uniformly at random from $\mathcal{H}$ makes every hash value equally likely for every item in the universe:

$$\Pr_{h \in \mathcal{H}}[h(x) = i] = \frac{1}{m} \quad \text{for all } x \in \mathcal{U} \text{ and all } i \in [m],$$

and *near-uniform* if the probability is bounded by at most $2/m$. A family $\mathcal{H}$ of hash functions is *universal* if, for any two items in the universe, the probability of collision is as small as possible, i.e.

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y) \mid x \neq y] \leq \frac{1}{m},$$

and *near-universal* if the probability of collision is at most $2/m$.

(a) [5pts] Describe a set of hash functions that is uniform but not (near-)universal.

(b) [5pts] Describe a set of hash functions that is universal but not (near-)uniform.