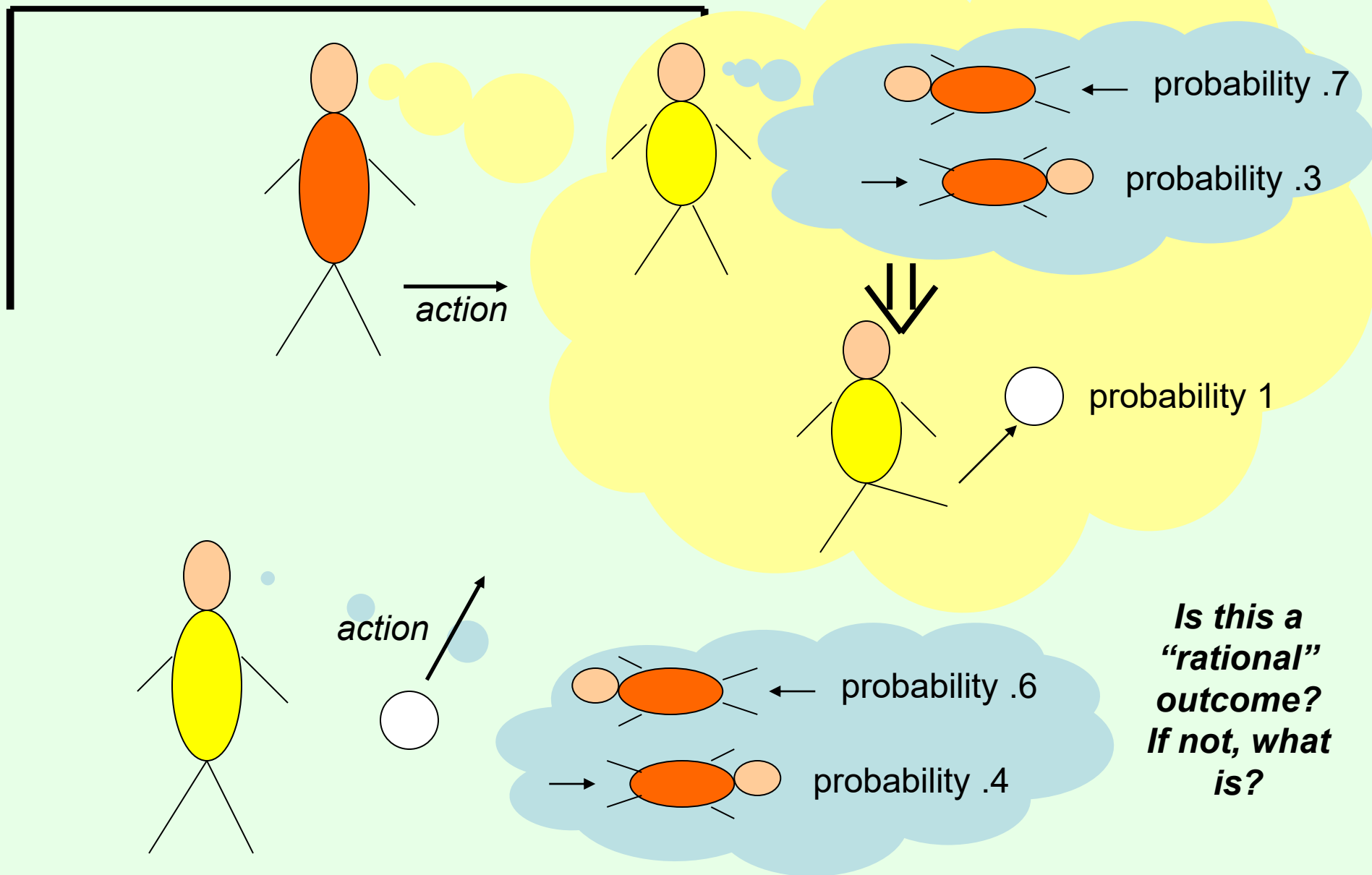


# Game Theory

Instructor: Vincent Conitzer

# Penalty kick example



***Is this a  
"rational"  
outcome?  
If not, what  
is?***

# Real-world security applications



*Milind Tambe's TEAMCORE group  
(USC→Harvard)*

## Airport security

Where should checkpoints, canine units,  
etc. be deployed?

## Federal Air Marshals

Which flights get a FAM?



## US Coast Guard

Which patrol routes should be followed?




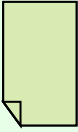


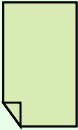

## Wildlife Protection


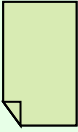

Where to patrol to catch poachers or find  
their snares?

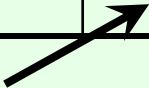


# Rock-paper-scissors

Column player aka.  
player 2  
(simultaneously)  
chooses a column



	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0



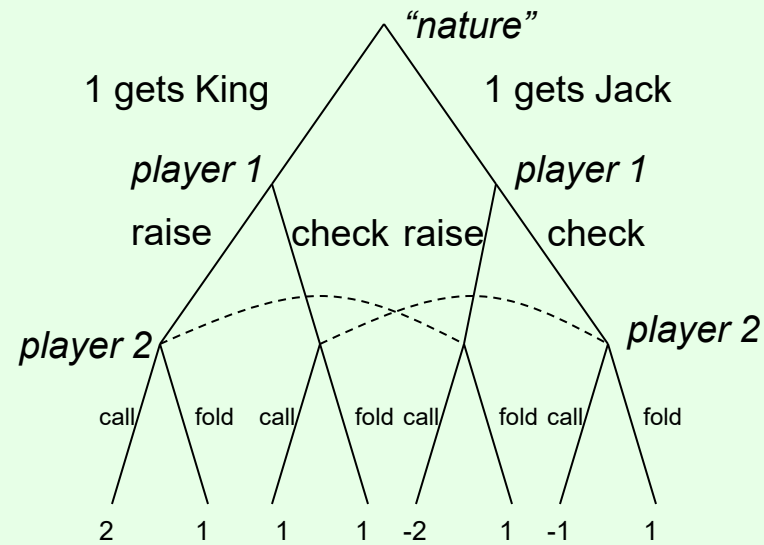
Row player  
aka. player 1  
chooses a row

A row or column is  
called an **action** or  
**(pure) strategy**

Row player's utility is always listed first, column player's second

**Zero-sum** game: the utilities in each entry sum to 0 (or a constant)  
Three-player game would be a 3D table with 3 utilities per entry, etc.

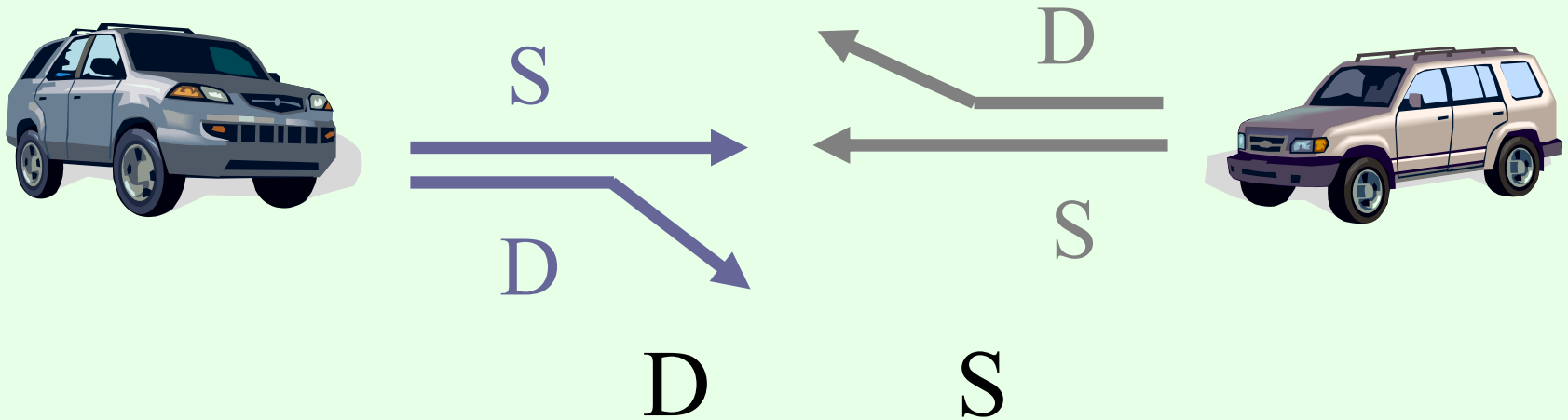
# A poker-like game



	cc	cf	fc	ff
rr	0, 0	0, 0	1, -1	1, -1
rc	.5, -.5	1.5, -1.5	0, 0	1, -1
cr	-.5, .5	-.5, .5	1, -1	1, -1
cc	0, 0	1, -1	0, 0	1, -1

# “Chicken”

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die



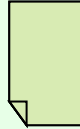
	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

not zero-sum

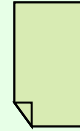
# “2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to  $2/3$  of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - $2/3$  of average = 33.33
  - A is closest ( $|50-33.33| = 16.67$ ), so A wins

# Rock-paper-scissors – Seinfeld variant



MICKEY: All right, rock beats paper!  
(Mickey smacks Kramer's hand for losing)  
KRAMER: I thought paper covered rock.  
MICKEY: Nah, rock flies right through paper.  
KRAMER: What beats rock?  
MICKEY: (looks at hand) Nothing beats rock.







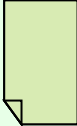
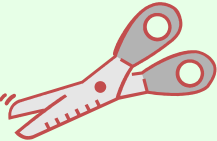
	Rock	Paper	Scissors
Rock	0, 0	1, -1	1, -1
Paper	-1, 1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0



# Dominance

- Player  $i$ 's strategy  $s_i$  **strictly dominates**  $s_i'$  if
  - for any  $s_{-i}$ ,  $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- $s_i$  **weakly dominates**  $s_i'$  if
  - for any  $s_{-i}$ ,  $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ ; and
  - for some  $s_{-i}$ ,  $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

$-i$  = "the player(s) other than  $i$ "

			
	0, 0	1, -1	1, -1
	-1, 1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

# Prisoner's Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

	confess	don't confess
confess	-2, -2	0, -3
don't confess	-3, 0	-1, -1

# “Should I buy an SUV?”

purchasing + gas cost



cost: 5



cost: 3

accident cost

cost: 5



cost: 5

cost: 8







cost: 2

cost: 5

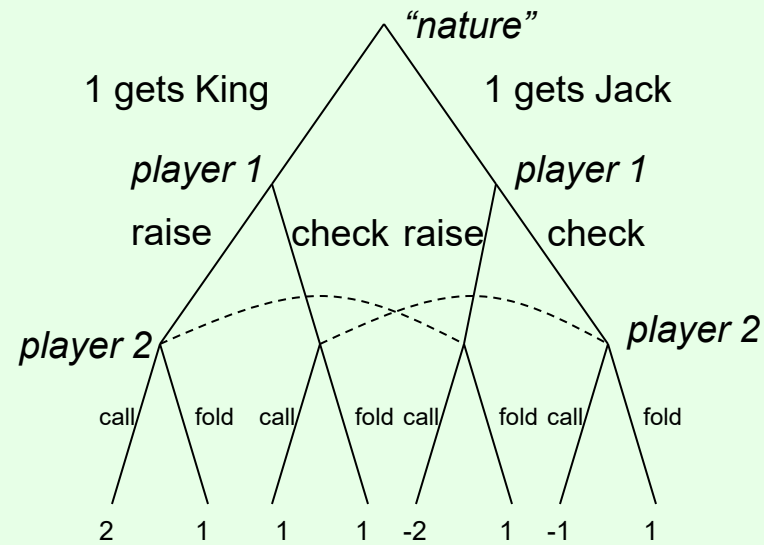


cost: 5



		
	-10, -10	-7, -11
	-11, -7	-8, -8

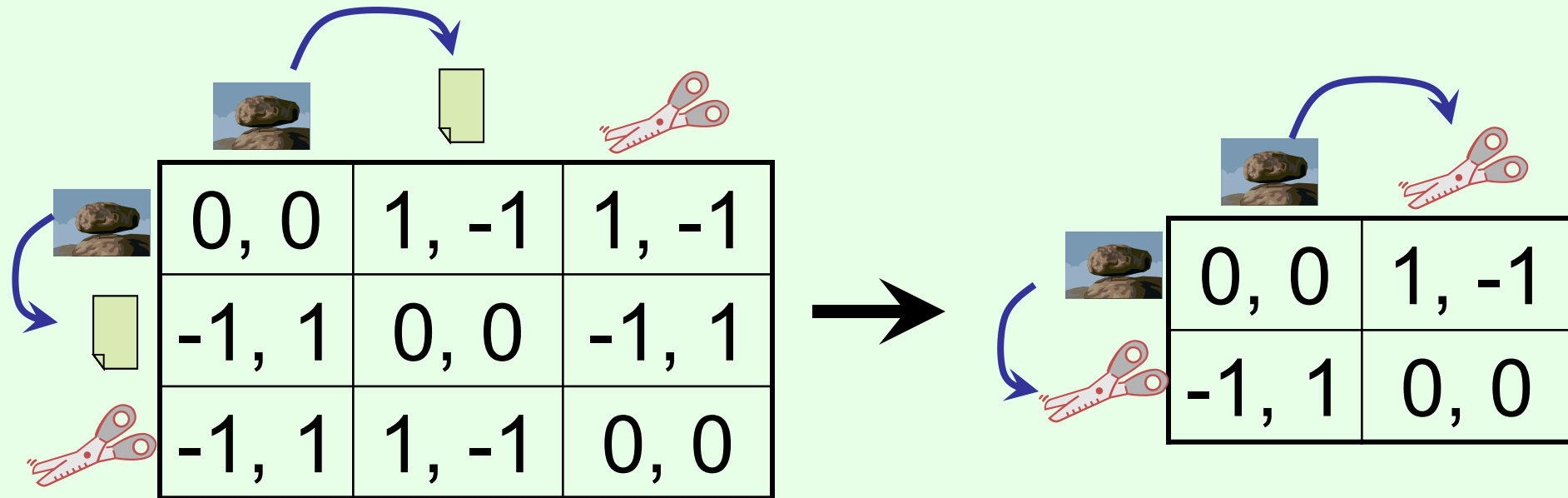
# Back to the poker-like game



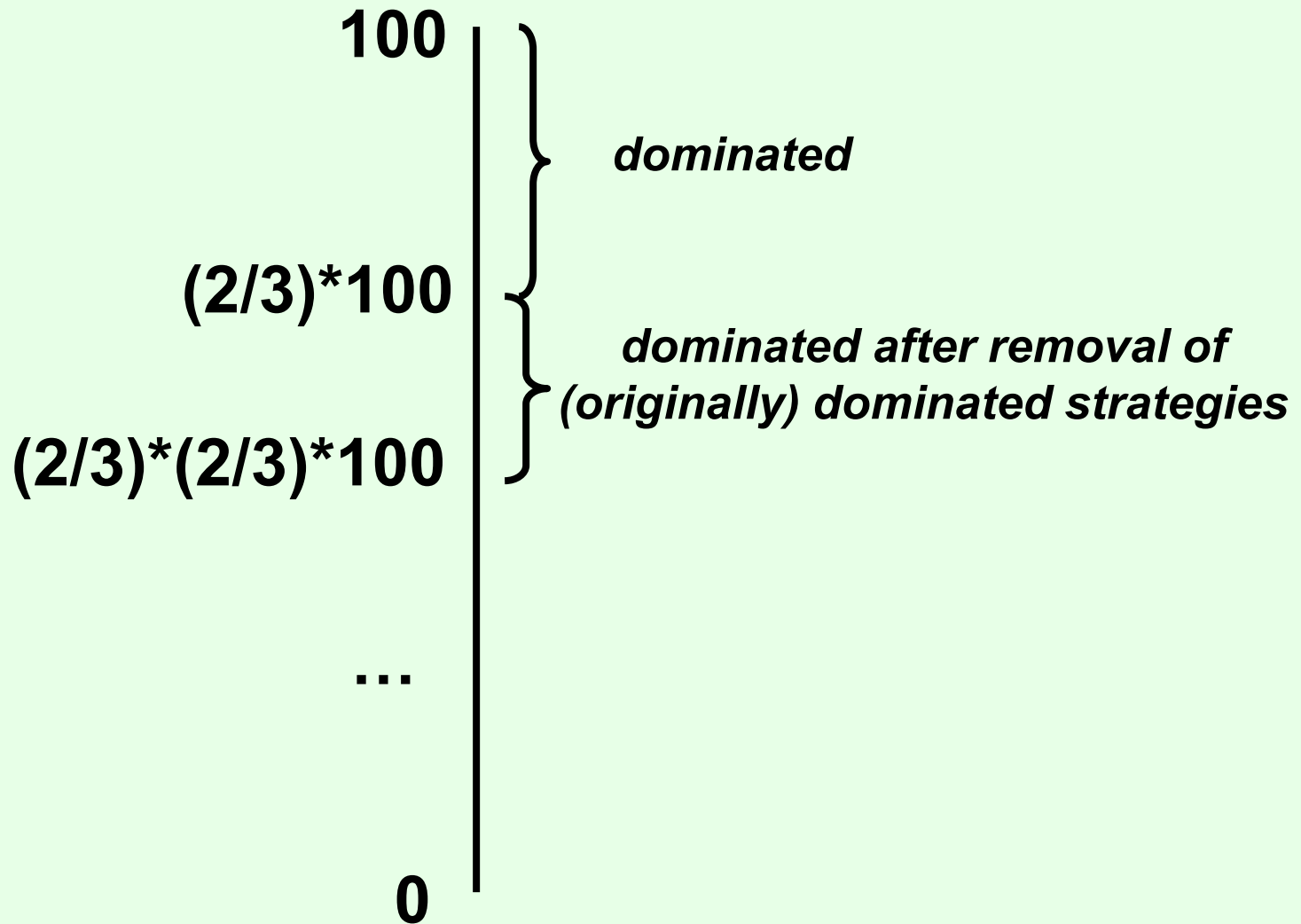
	cc	cf	fc	ff
rr	0, 0	<del>0, 0</del>	1, -1	<del>1, -1</del>
rc	.5, -.5	<del>1.5, -1.5</del>	0, 0	<del>1, -1</del>
cr	<del>-.5, .5</del>	<del>-.5, .5</del>	1, -1	<del>1, -1</del>
cc	0, 0	<del>1, -1</del>	0, 0	<del>1, -1</del>

# Iterated dominance


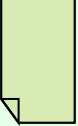

- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld's RPS:



# “2/3 of the average” game revisited



# Mixed strategies

- **Mixed strategy** for player  $i$  = **probability distribution** over player  $i$ 's (pure) strategies
- E.g.  $1/3$   ,  $1/3$   ,  $1/3$  
- Example of dominance by a mixed strategy:

$1/2$	3, 0	0, 0
$1/2$	0, 0	3, 0
	1, 0	1, 0

A blue bracket on the left side of the table groups the first two rows, with a curved arrow pointing from the bracket to the third row, indicating that the third row is dominated by the mixed strategy of the first two rows.

# Nash equilibrium [Nash 1950]




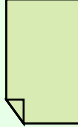


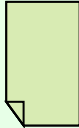

- A profile (= strategy for each player) so that no player wants to deviate

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- This game has another Nash equilibrium in mixed strategies...



# Rock-paper-scissors

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

- Any pure-strategy Nash equilibria?
- But it has a **mixed-strategy Nash equilibrium**:  
Both players put probability  $1/3$  on each action
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize

# Nash equilibria of “chicken” ...

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 **indifferent** between D and S
- Player 1's utility for playing D =  $-p^c_S$
- Player 1's utility for playing S =  $p^c_D - 5p^c_S = 1 - 6p^c_S$
- So we need  $-p^c_S = 1 - 6p^c_S$  which means  $p^c_S = 1/5$
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium:  $((4/5 D, 1/5 S), (4/5 D, 1/5 S))$ 
  - People may die! Expected utility  $-1/5$  for each player

# The presentation game



*Put effort into presentation (E)*

*Do not put effort into presentation (NE)*

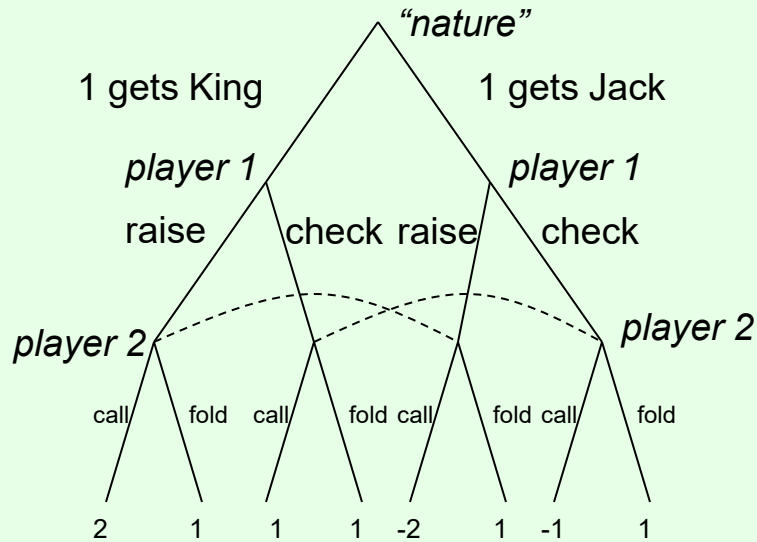
*Pay attention (A)*

*Do not pay attention (NA)*

$2, 2$	$-1, 0$
$-7, -8$	$0, 0$

- Pure-strategy Nash equilibria: (E, A), (NE, NA)
- Mixed-strategy Nash equilibrium:  
( $(4/5 E, 1/5 NE), (1/10 A, 9/10 NA)$ )
  - Utility  $-7/10$  for presenter, 0 for audience

# Back to the poker-like game, again



		$\frac{2}{3}$ cc	cf	$\frac{1}{3}$ fc	ff
$\frac{1}{3}$	rr	0, 0	0, 0	1, -1	1, -1
$\frac{2}{3}$	rc	.5, -.5	1.5, -1.5	0, 0	1, -1
	cr	-.5, .5	-1, -1	1, -1	1, -1
	cc	0, 0	1, -1	0, 0	1, -1

- To make player 1 indifferent between rr and rc, we need:  

$$\text{utility for rr} = 0 \cdot P(\text{cc}) + 1 \cdot (1 - P(\text{cc})) = .5 \cdot P(\text{cc}) + 0 \cdot (1 - P(\text{cc})) = \text{utility for rc}$$
 That is,  $P(\text{cc}) = \frac{2}{3}$
- To make player 2 indifferent between cc and fc, we need:  

$$\text{utility for cc} = 0 \cdot P(\text{rr}) + (-.5) \cdot (1 - P(\text{rr})) = -1 \cdot P(\text{rr}) + 0 \cdot (1 - P(\text{rr})) = \text{utility for fc}$$
 That is,  $P(\text{rr}) = \frac{1}{3}$

# Commitment

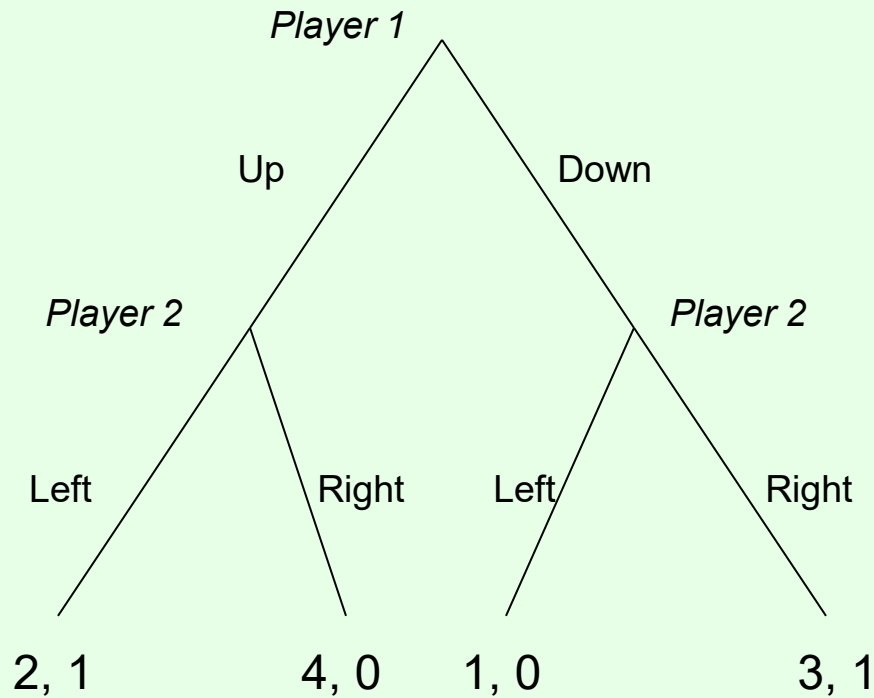
- Consider the following (normal-form) game:

2, 1	4, 0
1, 0	3, 1

- How should this game be played?
- Now suppose the game is played as follows:
  - Player 1 **commits** to playing one of the rows,
  - Player 2 observes the commitment and then chooses a column
- What is the optimal strategy for player 1?
- What if 1 can commit to a **mixed** strategy?

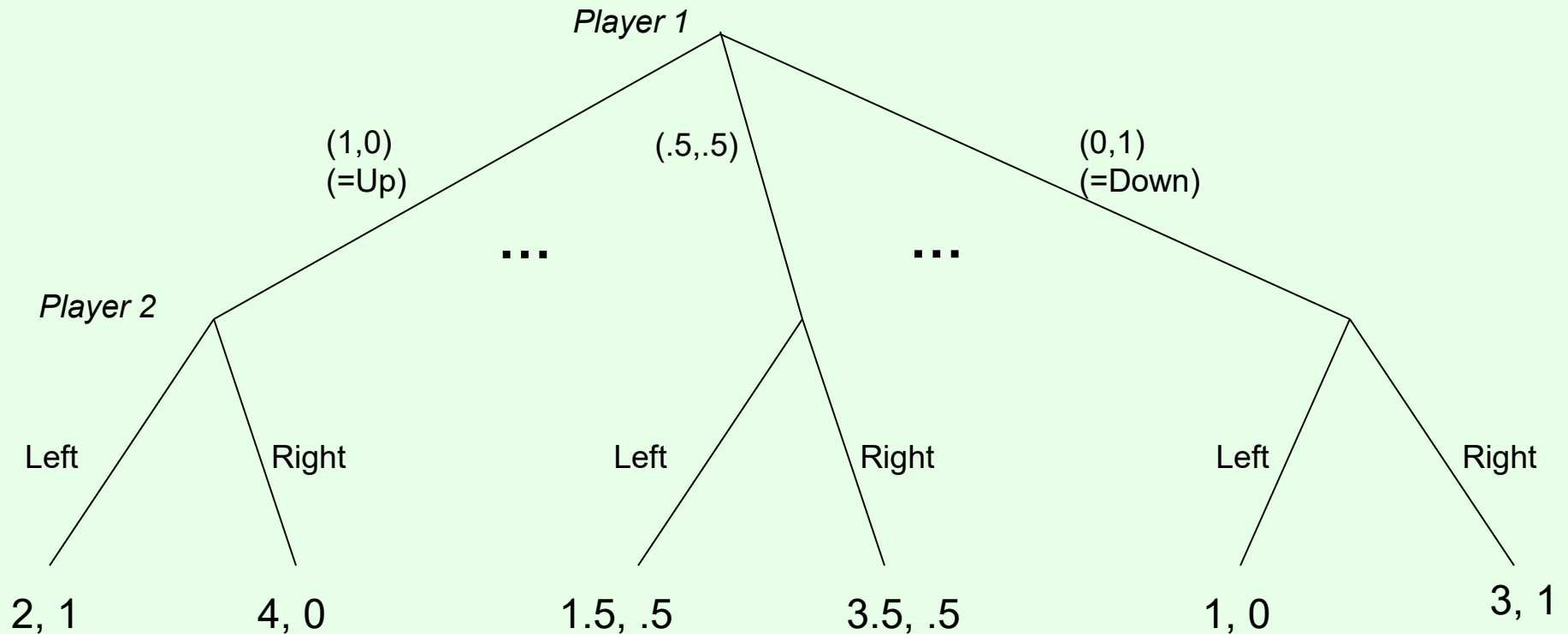
# Commitment as an extensive-form game

- For the case of committing to a pure strategy:



# Commitment as an extensive-form game

- For the case of committing to a mixed strategy:



- Infinite-size game; computationally impractical to reason with the extensive form here

# Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

- For **every** column  $c$  separately, we will solve separately for the best mixed row strategy (defined by  $\mathbf{p}_r$ ) that induces player 2 to play  $c$
- maximize  $\sum_r \mathbf{p}_r u_1(r, c)$
- subject to
  - for any  $c'$ ,  $\sum_r \mathbf{p}_r u_2(r, c) \geq \sum_r \mathbf{p}_r u_2(r, c')$
  - $\sum_r \mathbf{p}_r = 1$
- (May be infeasible, e.g., if  $c$  is strictly dominated)
- Pick the  $c$  that is best for player 1



# Visualization

	L	C	R
U	0,1	1,0	0,0
M	4,0	0,1	0,0
D	0,0	1,0	1,1

