Artificial Intelligence

Markov processes and Hidden Markov Models (HMMs)

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Motivation

- The Bayes nets we considered so far were static: they referred to a single point in time
  - E.g., medical diagnosis
- Agent needs to model how the world evolves
  - Speech recognition software needs to process speech over time
  - Artificially intelligent software assistant needs to keep track of user’s intentions over time
- … … …
Markov processes

- We have time periods \( t = 0, 1, 2, \ldots \)
- In each period \( t \), the world is in a certain state \( S_t \)
- The **Markov assumption**: given \( S_t \), \( S_{t+1} \) is independent of all \( S_i \) with \( i < t \)
  - \( P(S_{t+1} \mid S_1, S_2, \ldots, S_t) = P(S_{t+1} \mid S_t) \)
  - Given the current state, history tells us nothing more about the future

- Typically, all the CPTs are the same:

\[
P(S_{t+1} = j \mid S_t = i) = a_{ij} \quad \text{(stationarity assumption)}
\]
Weather example

• $S_t$ is one of \{s, c, r\} (sun, cloudy, rain)
• Transition probabilities:

\[
\begin{array}{ccc}
S & c & r \\
\text{s} & 0.1 & 0.2 & 0.3 \\
\text{c} & 0.4 & 0.3 & 0.5 \\
\text{r} & 0.3 & 0.1 & 0.6 \\
\end{array}
\]

not a Bayes net!

• Also need to specify an initial distribution $P(S_0)$
• Throughout, assume $P(S_0 = s) = 1$
• What is the probability that it rains two days from now?  \( P(S_2 = r) \)
• \( P(S_2 = r) = P(S_2 = r, S_1 = r) + P(S_2 = r, S_1 = s) + P(S_2 = r, S_1 = c) = .1* .3 + .6* .1 + .3* .3 = .18 \)
Weather example…

- What is the probability that it rains three days from now?
- Computationally inefficient way: \( P(S_3 = r) = P(S_3 = r, S_2 = r, S_1 = r) + P(S_3 = r, S_2 = r, S_1 = s) + \ldots \)
- For \( n \) periods into the future, need to sum over \( 3^{n-1} \) paths
Weather example...

- More efficient:
- \[ P(S_3 = r) = P(S_3 = r, S_2 = r) + P(S_3 = r, S_2 = s) + P(S_3 = r, S_2 = c) = P(S_3 = r | S_2 = r)P(S_2 = r) + P(S_3 = r | S_2 = s)P(S_2 = s) + P(S_3 = r | S_2 = c)P(S_2 = c) \]
- Only hard part: figure out \( P(S_2) \)
- Main idea: compute distribution \( P(S_1) \), then \( P(S_2) \), then \( P(S_3) \)
- Linear in number of periods!

*example on board*
Stationary distributions

- As time $t$ goes to infinity, "generally,” the distribution $P(S_t)$ will converge to a stationary distribution.

- A distribution given by probabilities $\pi_i$ (where $i$ is a state) is stationary if:

  \[ P(S_t = i) = \pi_i \]

  means that

  \[ P(S_{t+1} = i) = \pi_i \]

- Of course,

  \[ P(S_{t+1} = i) = \sum_j P(S_{t+1} = i, S_t = j) = \sum_j P(S_t = j) a_{ji} \]

- So, stationary distribution is defined by

  \[ \pi_i = \sum_j \pi_j a_{ji} \]
Computing the stationary distribution

- $\pi_s = 0.6\pi_s + 0.4\pi_c + 0.2\pi_r$
- $\pi_c = 0.3\pi_s + 0.3\pi_c + 0.5\pi_r$
- $\pi_r = 0.1\pi_s + 0.3\pi_c + 0.3\pi_r$
Restrictiveness of Markov models

- Are past and future really independent given current state?
- E.g., suppose that when it rains, it rains for at most 2 days

\[ S_1, S_2 \rightarrow S_2, S_3 \rightarrow S_3, S_4 \rightarrow S_4, S_5 \rightarrow \ldots \]

- Second-order Markov process
- Workaround: change meaning of “state” to events of last 2 days

\[ S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow \ldots \]

- Another approach: add more information to the state
- E.g., the full state of the world would include whether the sky is full of water
  - Additional information may not be observable
  - Blowup of number of states…
Hidden Markov models (HMMs)

- Same as Markov model, except we cannot see the state
- Instead, we only see an observation each period, which depends on the current state

Still need a transition model: $P(S_{t+1} = j \mid S_t = i) = a_{ij}$

Also need an observation model: $P(O_t = k \mid S_t = i) = b_{ik}$
Weather example extended to HMM

• Transition probabilities:

• Observation: labmate wet or dry
• $b_{sw} = .1$, $b_{cw} = .3$, $b_{rw} = .8$
HMM weather example: a question

• You have been stuck in the lab for three days (!)
• On those days, your labmate was dry, wet, wet, respectively
• What is the probability that it is now raining outside?
• $P(S_2 = r \mid O_0 = d, O_1 = w, O_2 = w)$
• By Bayes’ rule, really want to know $P(S_2, O_0 = d, O_1 = w, O_2 = w)$

$\begin{align*}
  b_{sw} & = .1 \\
  b_{cw} & = .3 \\
  b_{rw} & = .8
\end{align*}$
Solving the question

- Computationally efficient approach: first compute
  \[ P(S_1 = i, O_0 = d, O_1 = w) \]
  for all states \( i \)

- General case: solve for
  \[ P(S_t, O_0 = o_0, O_1 = o_1, \ldots, O_t = o_t) \]
  for \( t = 1, \) then \( t = 2, \ldots \) This is called monitoring

- \[ P(S_t, O_0 = o_0, O_1 = o_1, \ldots, O_t = o_t) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1}, O_0 = o_0, O_1 = o_1, \ldots, O_{t-1} = o_{t-1}) P(S_t | S_{t-1} = s_{t-1}) P(O_t = o_t | S_t) \]
You have been stuck in the lab for three days
On those days, your labmate was dry, wet, wet, respectively
What is the probability that two days from now it will be raining outside?

\[ P(S_4 = r \mid O_0 = d, O_1 = w, O_2 = w) \]
Predicting further out, continued...

- Want to know: $P(S_4 = r | O_0 = d, O_1 = w, O_2 = w)$
- Already know how to get: $P(S_2 | O_0 = d, O_1 = w, O_2 = w)$
- $P(S_3 = r | O_0 = d, O_1 = w, O_2 = w) = \sum_{s_2} P(S_3 = r, S_2 = s_2 | O_0 = d, O_1 = w, O_2 = w)$
- $\sum_{s_2} P(S_3 = r | S_2 = s_2)P(S_2 = s_2 | O_0 = d, O_1 = w, O_2 = w)$
- Etc. for $S_4$
- So: monitoring first, then straightforward Markov process updates

$b_{sw} = .1$
$b_{cw} = .3$
$b_{rw} = .8$
Integrating newer information

- You have been stuck in the lab for **four** days (!)
- On those days, your labmate was dry, wet, wet, dry respectively
- What is the probability that **two days ago** it was raining outside? \( P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d) \)
  - **Smoothing** or hindsight problem

\[
\begin{align*}
\text{b}_{sw} &= .1 \\
\text{b}_{cw} &= .3 \\
\text{b}_{rw} &= .8
\end{align*}
\]
Hindsight problem continued...

• Want: $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d)$

• “Partial” application of Bayes’ rule:
  $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d) =$
  $P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w) /$
  $P(O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$

• So really want to know $P(S_1, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$

- $b_{sw} = .1$
- $b_{cw} = .3$
- $b_{rw} = .8$
Hindsight problem continued…

\[ \begin{align*}
  \text{Want to know } & P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w) \\
  \text{P}(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w) = & \quad P(S_1 = r \mid O_0 = d, O_1 = w) \cdot P(O_2 = w, O_3 = d \mid S_1 = r) \\
  \text{Already know how to compute } & P(S_1 = r \mid O_0 = d, O_1 = w) \\
  \text{Just need to compute } & P(O_2 = w, O_3 = d \mid S_1 = r)
\end{align*} \]

- \[ b_{sw} = 0.1 \]
- \[ b_{cw} = 0.3 \]
- \[ b_{rw} = 0.8 \]
Hindsight problem continued…

- Just need to compute $P(O_2 = w, O_3 = d \mid S_1 = r)$
- $P(O_2 = w, O_3 = d \mid S_1 = r) = \\
  \sum_{s_2} P(S_2 = s_2, O_2 = w, O_3 = d \mid S_1 = r) = \\
  \sum_{s_2} P(S_2 = s_2 \mid S_1 = r) P(O_2 = w \mid S_2 = s_2) P(O_3 = d \mid S_2 = s_2)$
- First two factors directly in the model; last factor is a “smaller” problem of the same kind
- Use dynamic programming, backwards from the future
  - Similar to forwards approach from the past

\[b_{sw} = .1\]
\[b_{cw} = .3\]
\[b_{rw} = .8\]
Backwards reasoning in general

• Want to know $P(O_{k+1} = o_{k+1}, \ldots, O_t = o_t | S_k)$

• First compute

$$P(O_t = o_t | S_{t-1}) = \sum_{s_t} P(S_t = s_t | S_{t-1})P(O_t = o_t | S_t = s_t)$$

• Then compute

$$P(O_t = o_t, O_{t-1} = o_{t-1} | S_{t-2}) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1} | S_{t-2})P(O_{t-1} = o_{t-1} | S_{t-1} = s_{t-1})P(O_t = o_t | S_{t-1} = s_{t-1})$$

• Etc.
Variable elimination

- Because all of this is inference in a Bayes net, we can also just do variable elimination

\[ P(S_3 = r, O_1 = d, O_2 = w, O_3 = w) = \]
\[ \sum_{s_2} \sum_{s_1} P(S_1 = s_1) P(O_1 = d | S_1 = s_1) P(S_2 = s_2 | S_1 = s_1) P(O_2 = w | S_2 = s_2) P(S_3 = r | S_2 = s_2) P(O_3 = w | S_3 = r) \]

- It’s a tree, so variable elimination works well
Dynamic Bayes Nets

• So far assumed that each period has one variable for state, one variable for observation
• Often better to divide state and observation up into multiple variables

edges both within a period, and from one period to the next...
Some interesting things we skipped

• Finding the most likely sequence of states, given observations
  – Not necessary equal to the sequence of most likely states! (example?)
  – Viterbi algorithm
    • Key idea: for each period t, for every state, keep track of most likely sequence to that state at that period, given evidence up to that period

• Continuous variables

• Approximate inference methods
  – Particle filtering