Artificial Intelligence

More search: When the path to the solution doesn't matter

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Search where the path doesn't matter

- So far, looked at problems where the path was the solution
  - Traveling on a graph
  - Eights puzzle
- However, in many problems, we just want to find a goal state
  - Doesn't matter how we get there

Queens puzzle

- Place eight queens on a chessboard so that no two attack each other

				Q			
		Q					
Q							
						Q	
	Q						
							Q
					Q		
			Q				

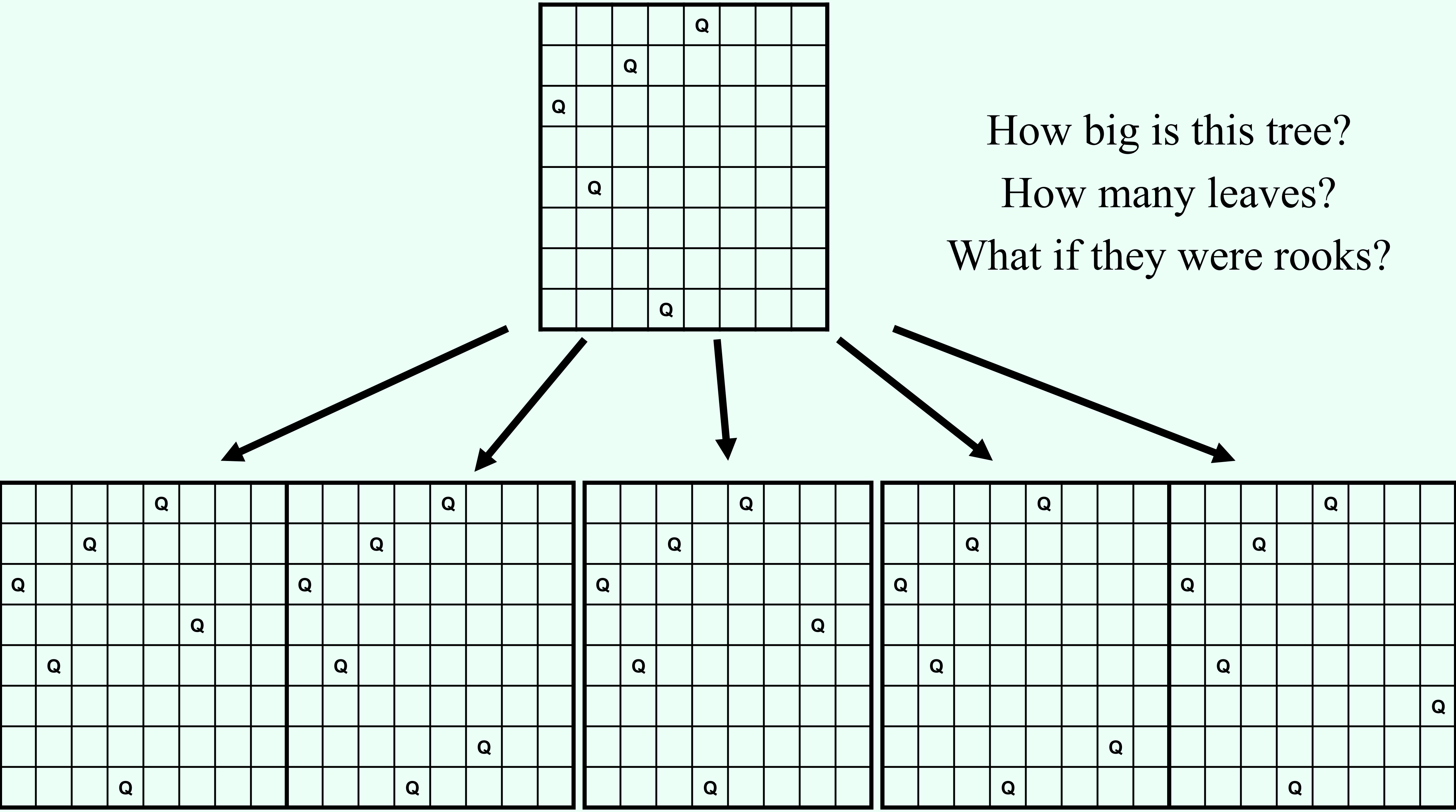
Search formulation of the queens puzzle

- **Successors**: all valid ways of placing additional queen on the board; **goal**: eight queens placed

How big is this tree?

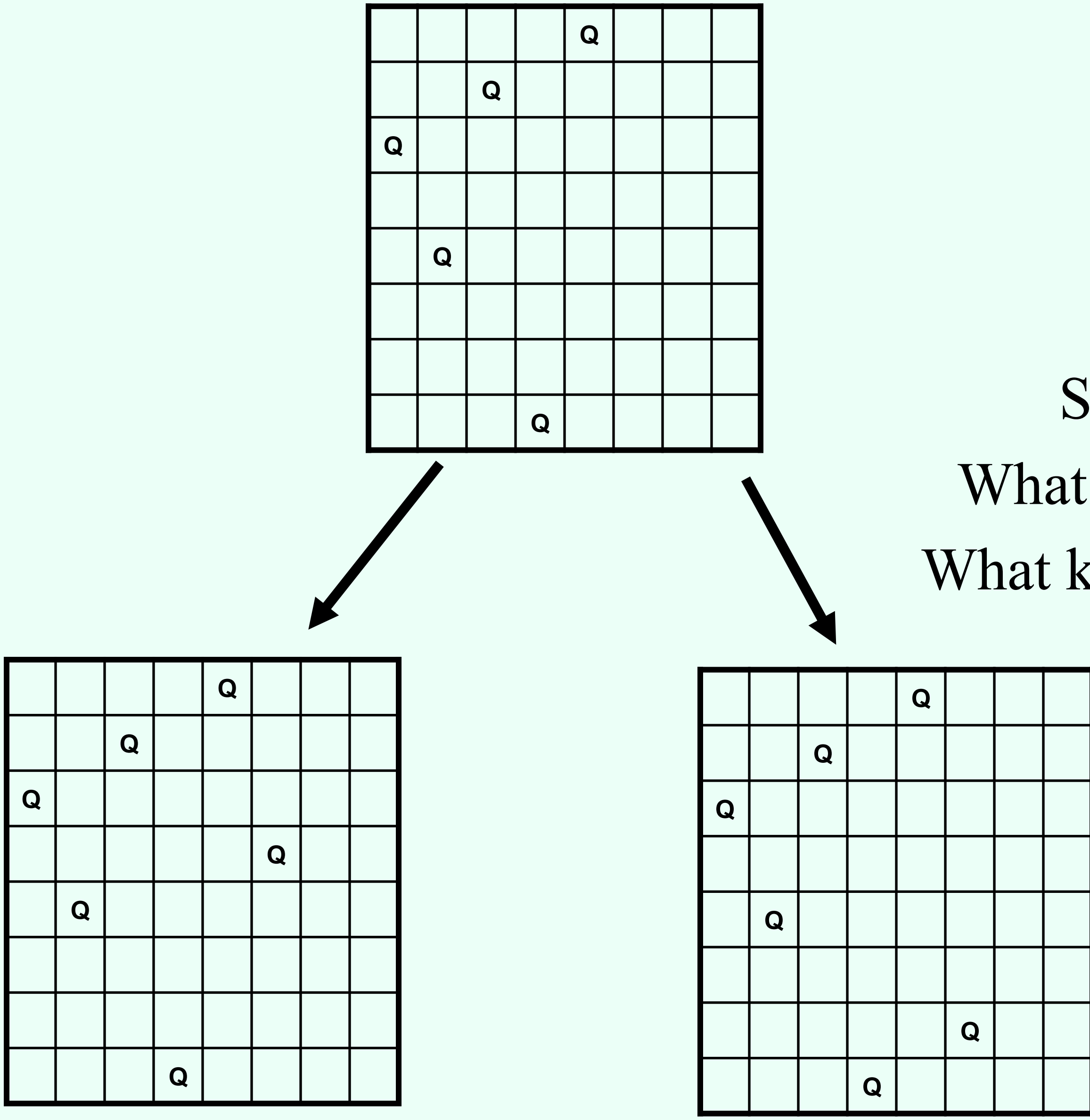
How many leaves?

What if they were rooks?



Search formulation of the queens puzzle

- **Successors**: all valid ways of placing a queen in the next column; **goal**: eight queens placed



Search tree size?

What if they were rooks?

What kind of search is best?

Constraint satisfaction problems (CSPs)

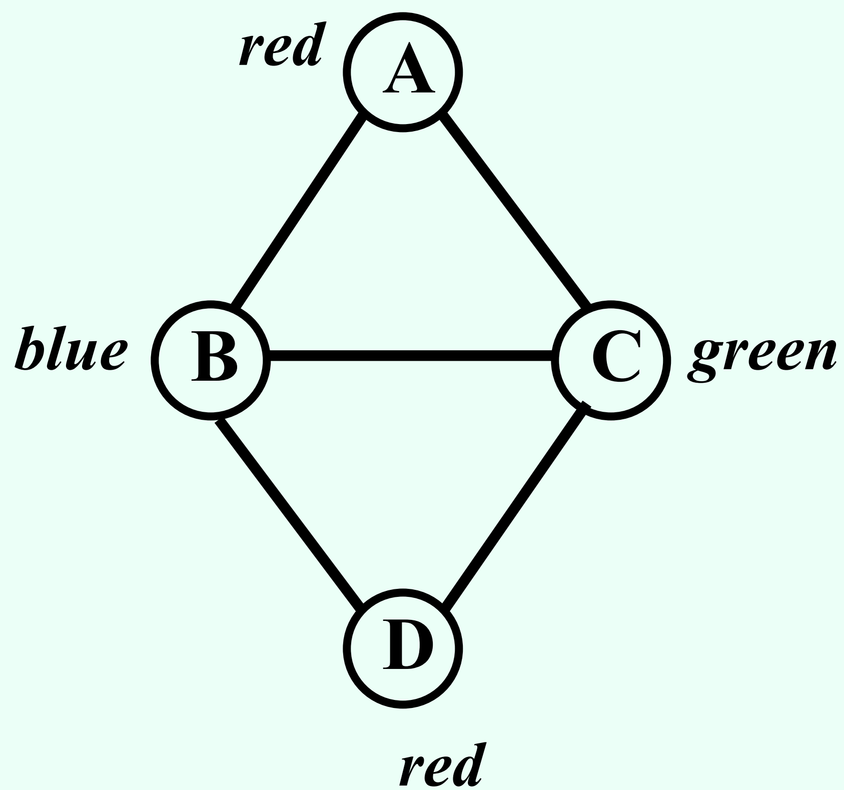
- Defined by:
  - A set of variables $x_1, x_2, \ldots, x_n$
  - A domain D_i for each variable x_i
  - Constraints $c_1, c_2, \ldots, c_m$
- A constraint is specified by
  - A subset (often, two) of the variables
  - All the allowable joint assignments to those variables
- Goal: find a complete, consistent assignment
- Queens problem: (other examples in next slides)
  - x_i in {1, …, 8} indicates in which row in the ith column to place a queen
  - For example, constraint on x_1 and x_2: {(1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,4), (2,5), …, (3,1), (3,5), … …}

Meeting scheduling

- Meetings A, B, C, … need to be scheduled on M, Tu, W, Th, F
- A and B cannot be scheduled on the same day
- B needs to be scheduled at least two days before C
- C cannot be scheduled on Th or F
- Etc.
- How do we model this as a CSP?

Graph coloring

- Fixed number of colors; no two adjacent nodes can share a color



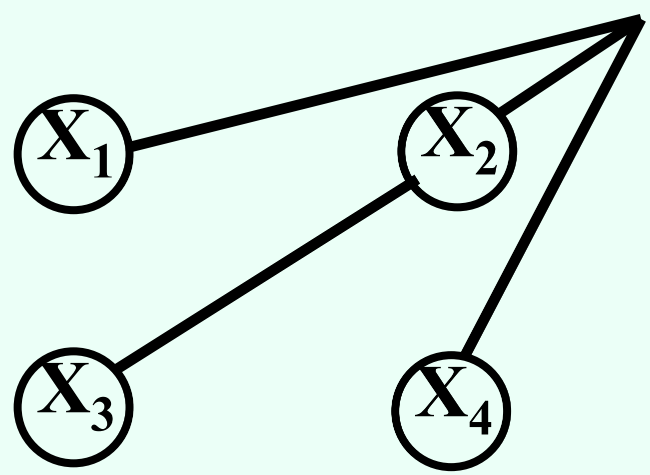
Satisfiability

- Formula in conjunctive normal form:

  (x_1 OR x_2 OR NOT(x_4)) AND (NOT(x_2) OR NOT(x_3)) AND …

  – Label each variable x_j as true or false so that the formula becomes true

Constraint hypergraph: each hyperedge represents a constraint



Cryptarithmetic puzzles

$$
\begin{array}{r}
TWO \\
TWO\ + \\
\hline
FOUR
\end{array}
$$

E.g., setting F = 1, O = 4, R = 8, T = 7, W = 3, U = 6 gives 734+734=1468

Cryptarithmetic puzzles…

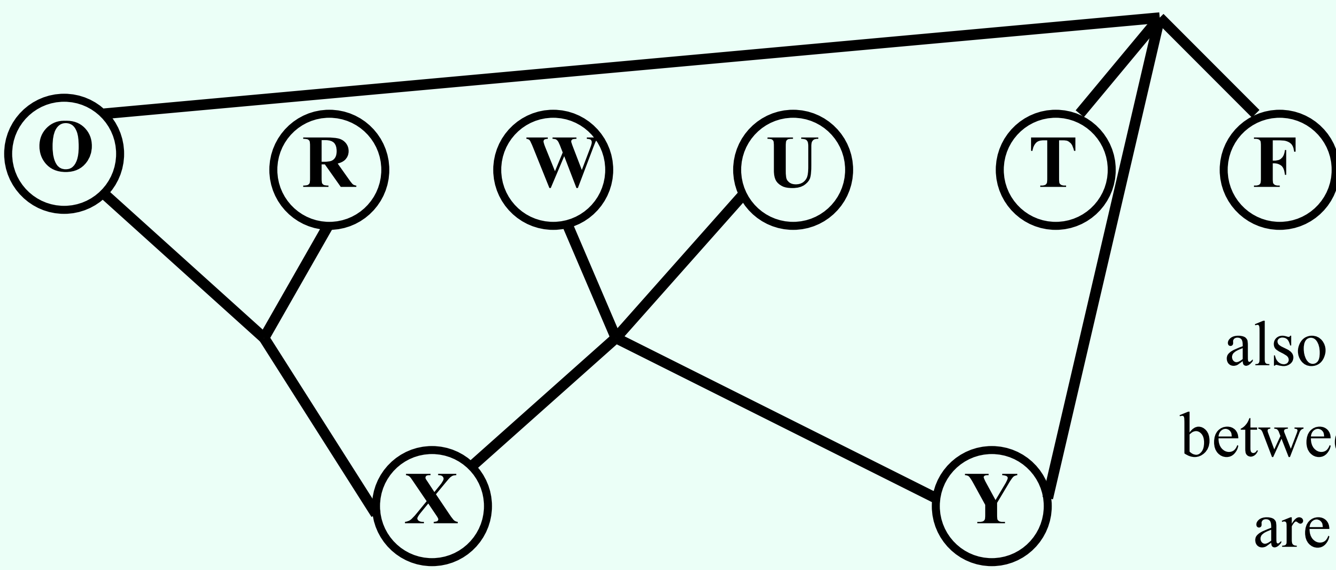
$$
\begin{array}{r}
T\ W\ O \\
T\ W\ O\ + \\
\hline
F\ O\ U\ R
\end{array}
$$

Trick: introduce auxiliary variables X, Y

O + O = 10X + R

W + W + X = 10Y + U

T + T + Y = 10F + O



What would the search tree look like?

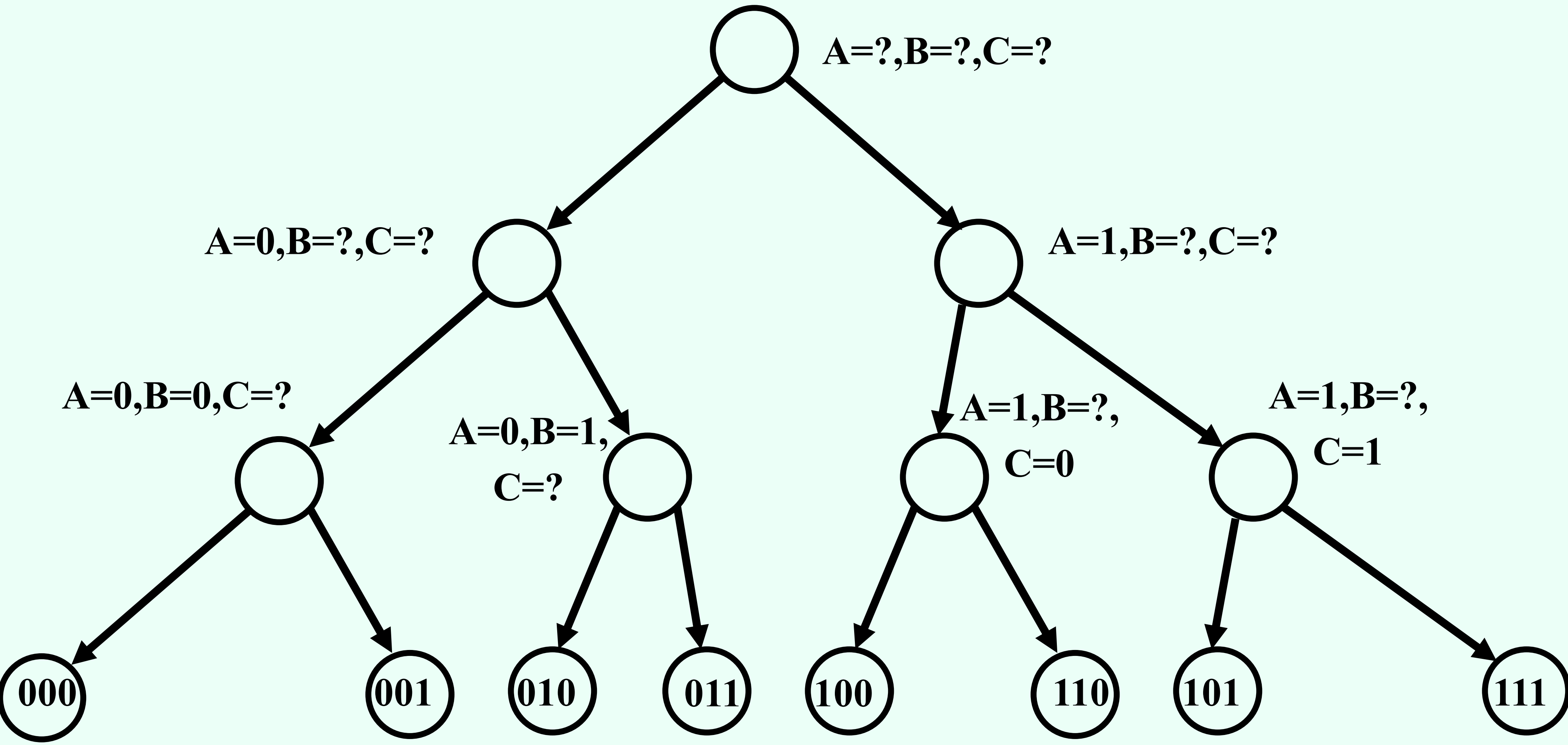
also need pairwise constraints between original variables if they are supposed to be different

Generic approaches to solving CSPs

- State: some variables assigned, others not assigned
- Naïve successors definition: any way of assigning a value to an unassigned variable results in a successor
  - Can check for consistency when expanding
  - How many leaves do we get in the worst case?
- CSPs satisfy commutativity: order in which actions applied does not matter
- Better idea: only consider assignments for a single variable at a time
  - How many leaves?

Choice of variable to branch on is still flexible!

- Do not always need to choose same variable at same level
- Each of variables A, B, C takes values in {0,1}



- Can you prove that this never increases the size of the tree?

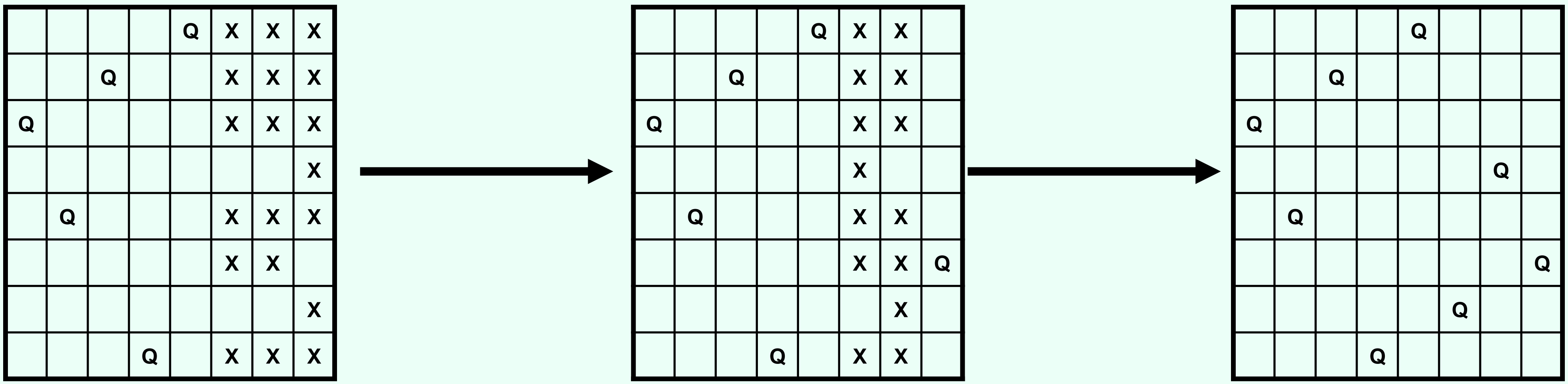
A generic recursive search algorithm

(*assignment* is a partial assignment)

- **Search(*assignment*, *constraints*)**
- If *assignment* is complete, return it
- Choose an unassigned variable *x*
- For every value *v* in *x*'s domain, if setting *x* to *v* in *assignment* does not violate *constraints:*
  - Set *x* to *v* in *assignment*
  - *result* := Search(*assignment*, *constraints*)
  - If *result* != *failure* return *result*
  - Unassign *x* in *assignment*
- Return *failure*

Keeping track of remaining possible values

- For every variable, keep track of which values are still possible



only one possibility for last column; might as well fill in

now only one left for other two columns

done!
(no real branching needed!)

- General heuristic: branch on variable with fewest values remaining

Arc consistency

- Take two variables connected by a constraint
- Is it true that for **every** remaining value *d* of the first variable, there exists **some** value *d'* of the other variable so that the constraint is satisfied?
  - If so, we say the arc from the first to the second variable is consistent
  - If not, can remove the value *d*
- General concept: constraint propagation

Q				x			x
				x			
		Q		x			x
				x			
				x	Q		x
				x			

Consider cryptarithmetic puzzle again…

Is the arc from the fifth to the eighth column consistent?

What about the arc from the eighth to the fifth?

Maintaining arc consistency

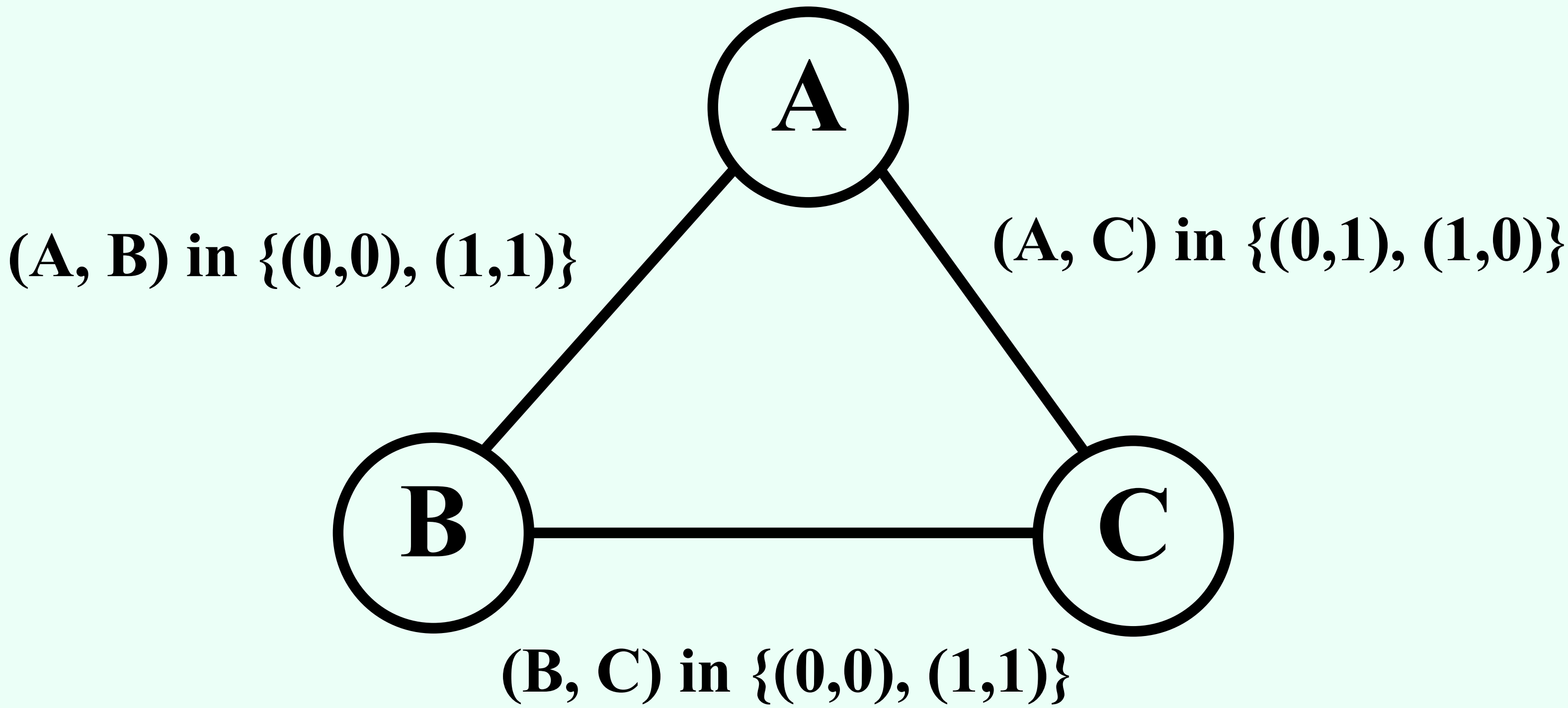
- Maintain a queue *Q* of all ordered pairs of variables with a constraint (arcs) that need to be checked
- Take a pair *(x, y)* from the queue
- For every value *v* in *x*'s domain, check if there is some value *w* in *y*'s domain so that *x=v, y=w* is consistent
  - If not, remove *v* from *x*'s domain
- If anything was removed from x's domain, add **every** arc *(z, x)* to *Q*
- Continue until *Q* is empty

- Runtime?
- *n* variables, *d* values per domain
- $O(n^2)$ arcs;
- each arc is added to the queue at most *d* times;
- consistency of an arc can be checked with d^2 lookups in the constraint's table;
- so $O(n^2d^3)$ lookups
- Can we do better?

Maintaining arc consistency (2)

- For every arc *(x, y)*, for every value *v* for *x*, maintain the number *n((x, y), v)* of remaining values for *y* that are consistent with *x=v*
- Every time that some *n((x, y), v) = 0*,
  - remove *v* from *x*'s domain;
  - for every arc *(z, x)*, for every value *w* for *z*, if *(x=v, z=w)* is consistent with the constraint, reduce *n((z, x), w)* by *1*
- Runtime:
  - for every arc *(z, x)* (n^2 of them), a value is removed from *x*'s domain at most *d* times;
  - each time we have to check for at most *d* of *z*'s values whether it is consistent with the removed value for *x*;
  - so $O(n^2d^2)$ lookups

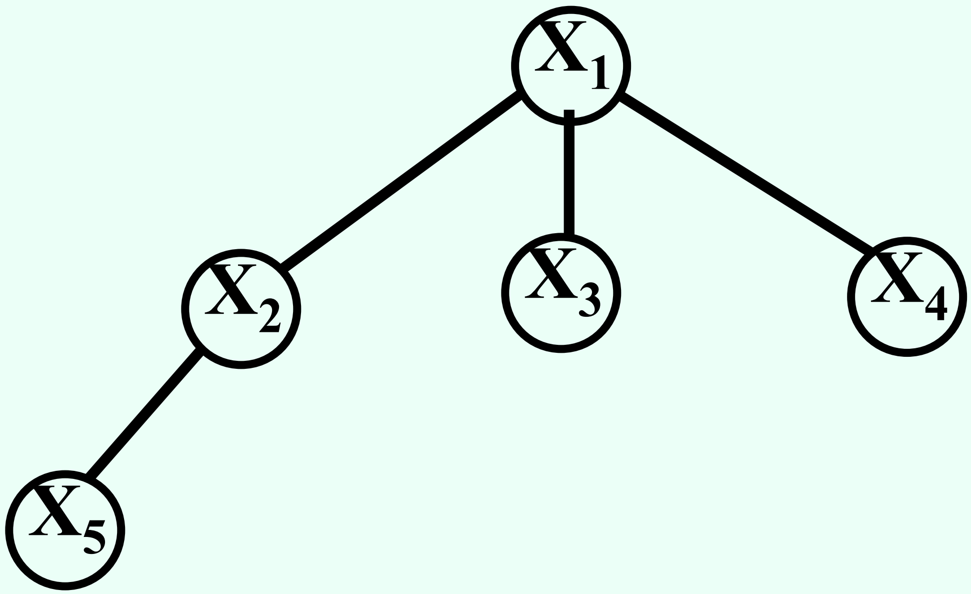
An interesting example



- A = B, B = C, C ≠ A – obviously inconsistent
  - ~ Moebius band
- However, arc consistency cannot eliminate anything

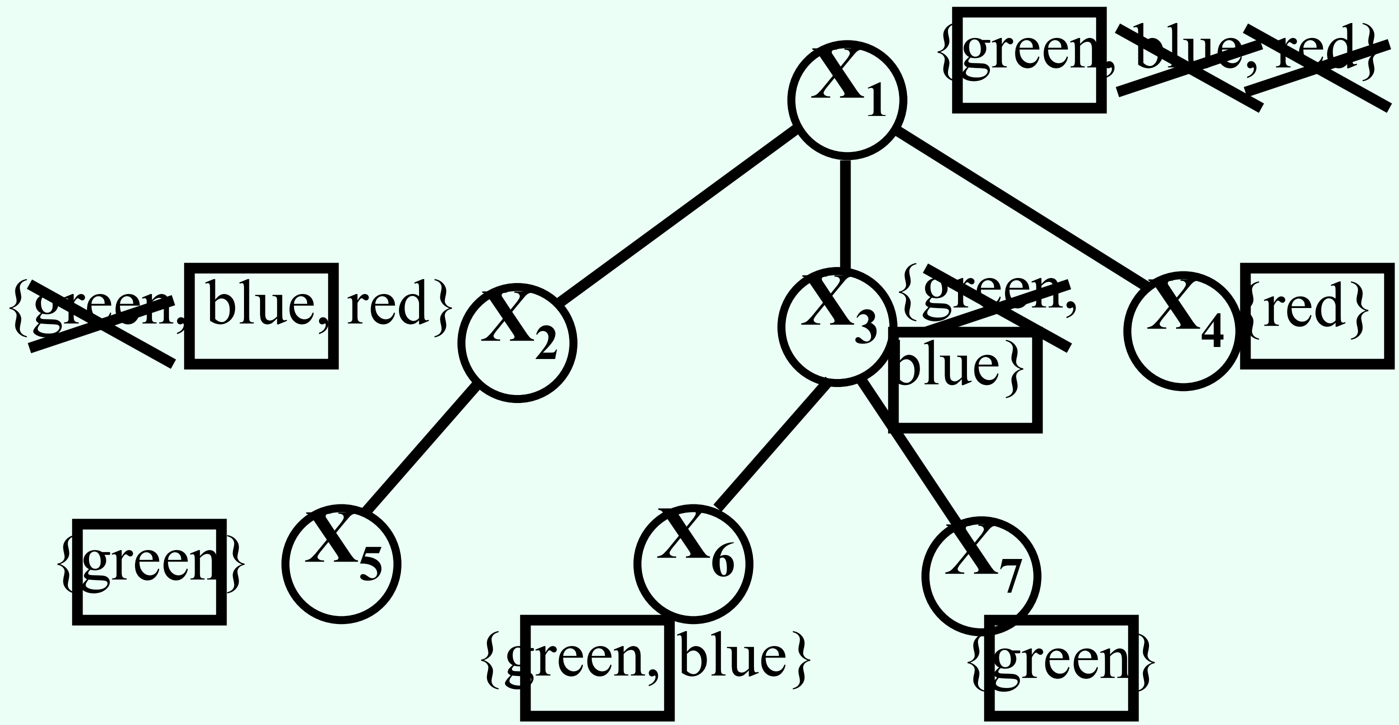
Tree-structured constraint graphs

- Suppose we only have pairwise constraints and the graph is a tree (or forest = multiple disjoint trees)



- Dynamic program for solving this (linear in #variables):
  - Starting from the leaves and going up, for each node *x*, compute all the values for *x* such that the subtree rooted at *x* can be solved
    - Equivalently: apply arc consistency from each parent to its children, starting from the bottom
  - If no domain becomes empty, once we reach the top, easy to fill in solution

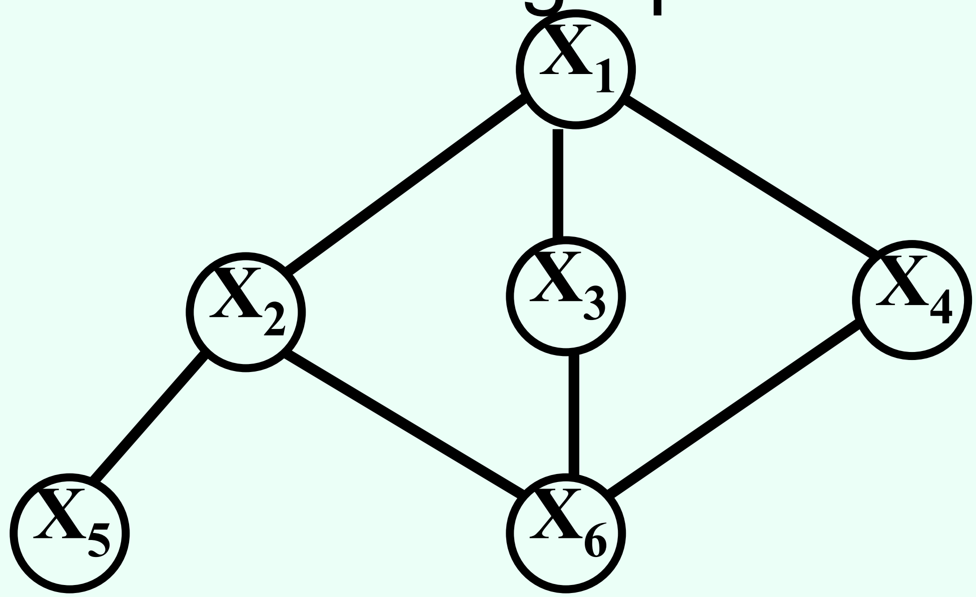
Example: graph coloring with limited set of colors per node



- Stage 1: moving upward, cross out the values that cannot work with the subtree below that node
- Stage 2: if a value remains at the root, there is a solution: go downward to pick a solution

Generalizations of the tree-based approach

- What if our constraint graph is "almost" a tree?



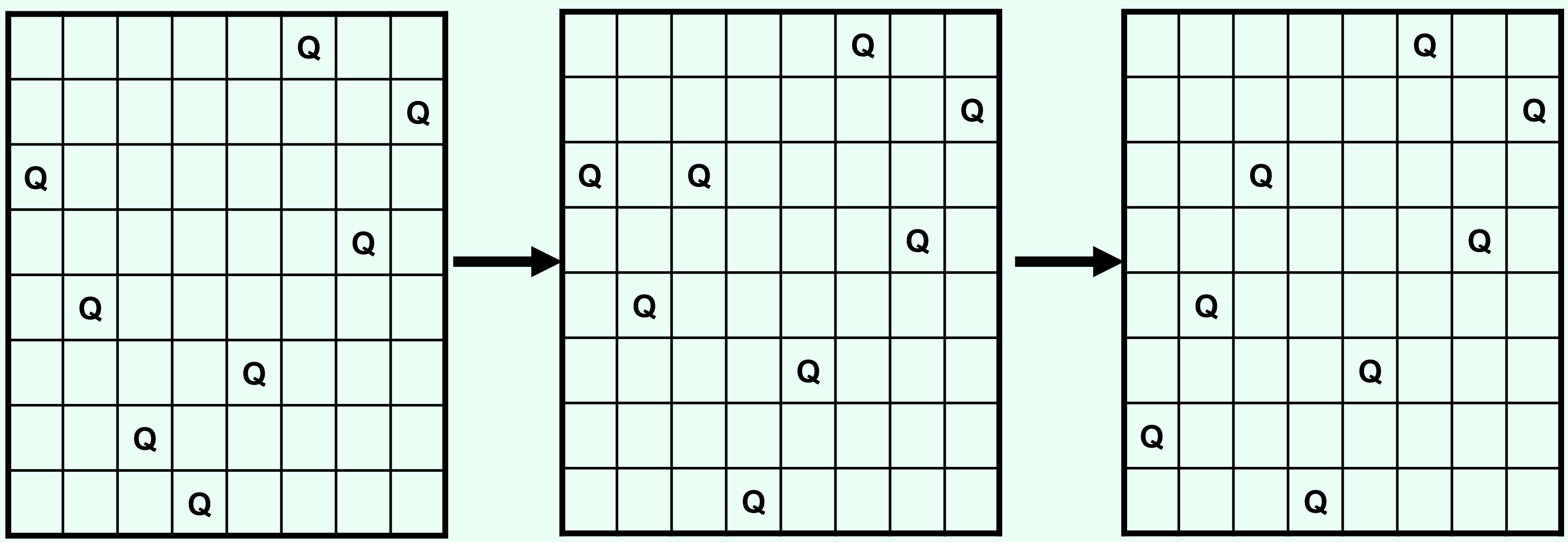
- A cycle cutset is a set of variables whose removal results in a tree (or forest)
  - E.g. $\{X_1\}$, $\{X_6\}$, $\{X_2, X_3\}$, $\{X_2, X_4\}$, $\{X_3, X_4\}$
- Simple algorithm: for every internally consistent assignment to the cutset, solve the remaining tree as before (runtime?)
- Graphs of bounded treewidth can also be solved in polynomial time (won't define these here)

A different approach: optimization

- Let's say every way of placing 8 queens on a board, one per column, is feasible
- Now we introduce an objective: minimize the number of pairs of queens that attack each other
  - More generally, minimize the number of violated constraints
- Pure optimization

Local search: hill climbing

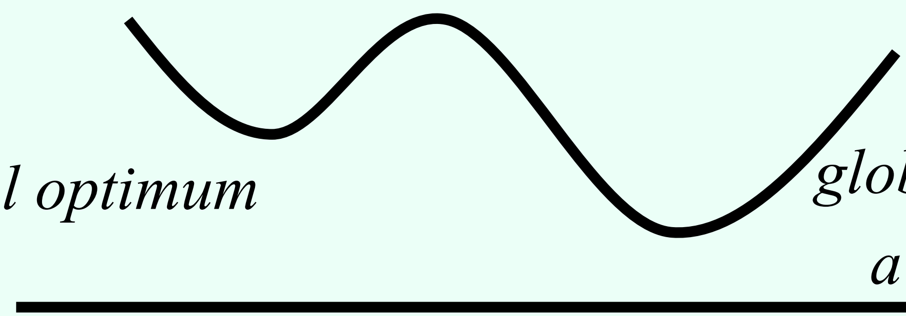
- Start with a complete state
- Move to successor with best (or at least better) objective value
  - Successor: move one queen within its column



no more improvements

4 attacking pairs *3 attacking pairs* *2 attacking pairs*

- Local search can get stuck in a local optimum



local optimum

global optimum (also a local optimum)

Avoiding getting stuck with local search

- Random restarts: if your hill-climbing search fails (or returns a result that may not be optimal), restart at a random point in the search space
  - Not always easy to generate a random state
  - Will **eventually** succeed (why?)
- Simulated annealing:
  - Generate a random successor (possibly worse than current state)
  - Move to that successor with some probability that is sharply decreasing in the badness of the state
  - Also, over time, as the "temperature decreases," probability of bad moves goes down

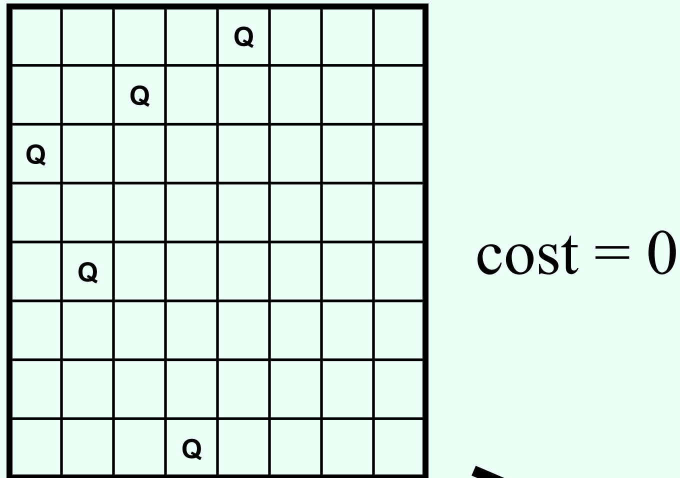
Constraint optimization

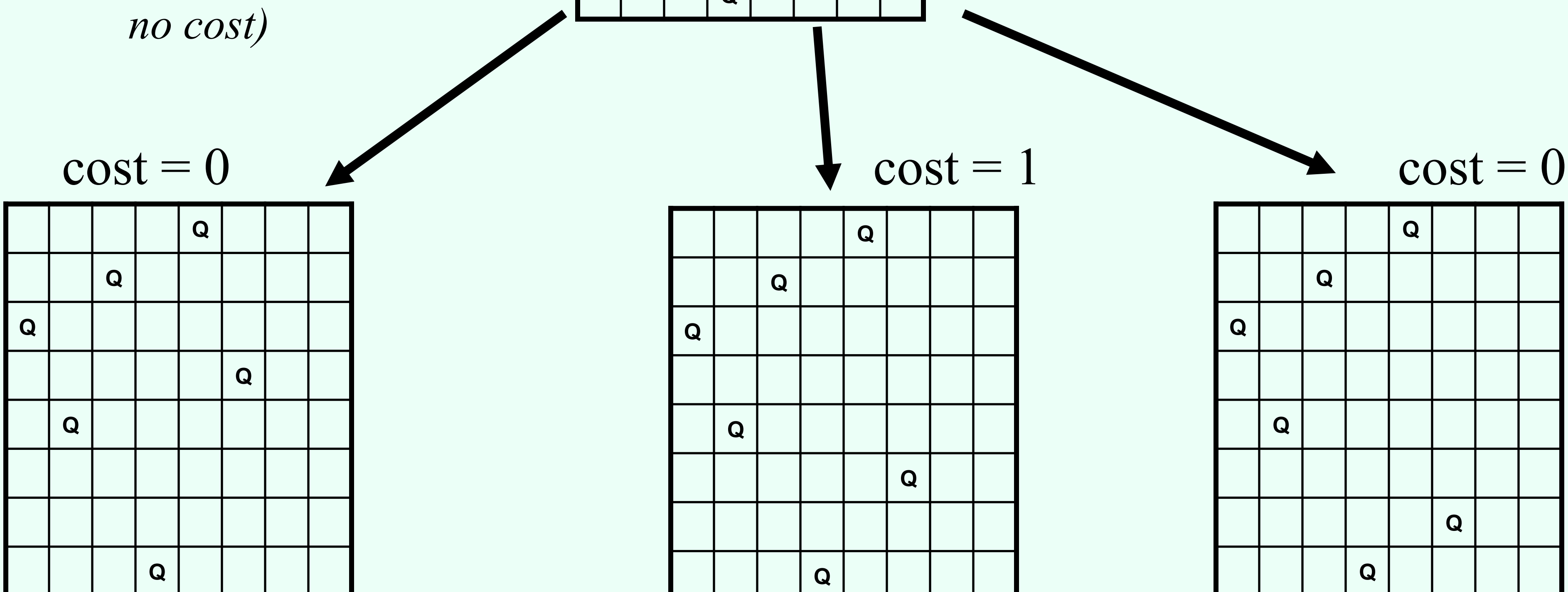
- Like a CSP, but with an objective
  - E.g., minimize number of violated constraints
  - Another example: no two queens can be in the same row or column (hard constraint), minimize number of pairs of queens attacking each other diagonally (objective)
- Can use all our techniques from before: heuristics, A*, IDA*, …
- Also popular: depth-first branch-and-bound
  - Like depth-first search, except do not stop when first feasible solution found; keep track of best solution so far
  - Given admissible heuristic, do not need to explore nodes that are worse than best solution found so far

Minimize #violated diagonal constraints

- **Cost of a node**: #violated diagonal constraints so far
- No heuristic

(matter of definition; could just as well say that violated constraints so far is the heuristic and interior nodes have no cost)





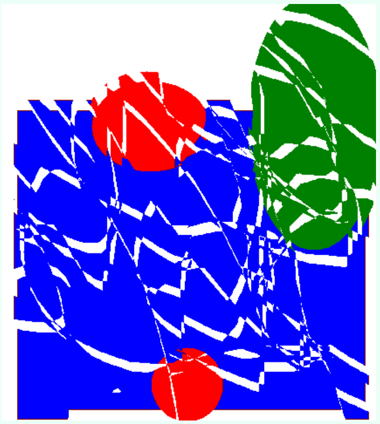
Depth first branch and bound will find a suboptimal solution here first (no way to tell at this point this is worse than right node)

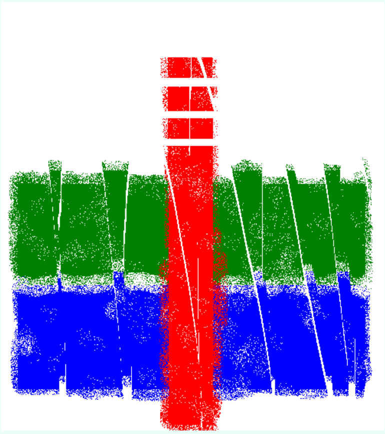
A* (=uniform cost here), IDA* (=iterative lengthening here) will **never** explore this node

Optimal solution is down here (cost 0)

Linear programs: example

- We make reproductions of two paintings





- Painting 1 sells for \$30, painting 2 sells for \$20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Solving the linear program graphically

maximize 3x + 2y

subject to

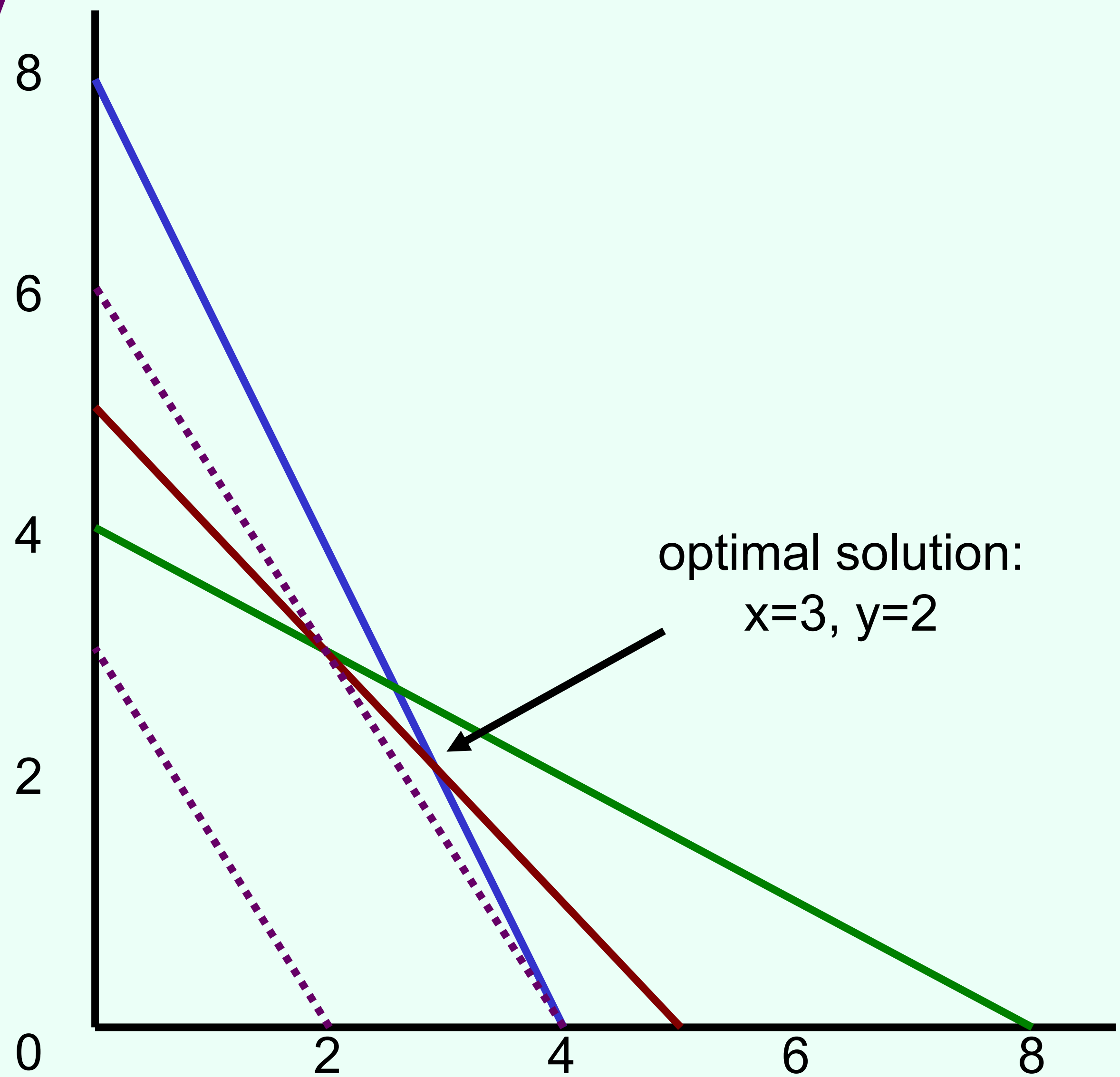
4x + 2y ≤ 16

x + 2y ≤ 8

x + y ≤ 5

x ≥ 0

y ≥ 0



Modified LP

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Optimal solution: $x = 2.5$, $y = 2.5$

Solution value $= 7.5 + 5 = 12.5$

Half paintings?

Integer (linear) program

maximize 3x + 2y

subject to

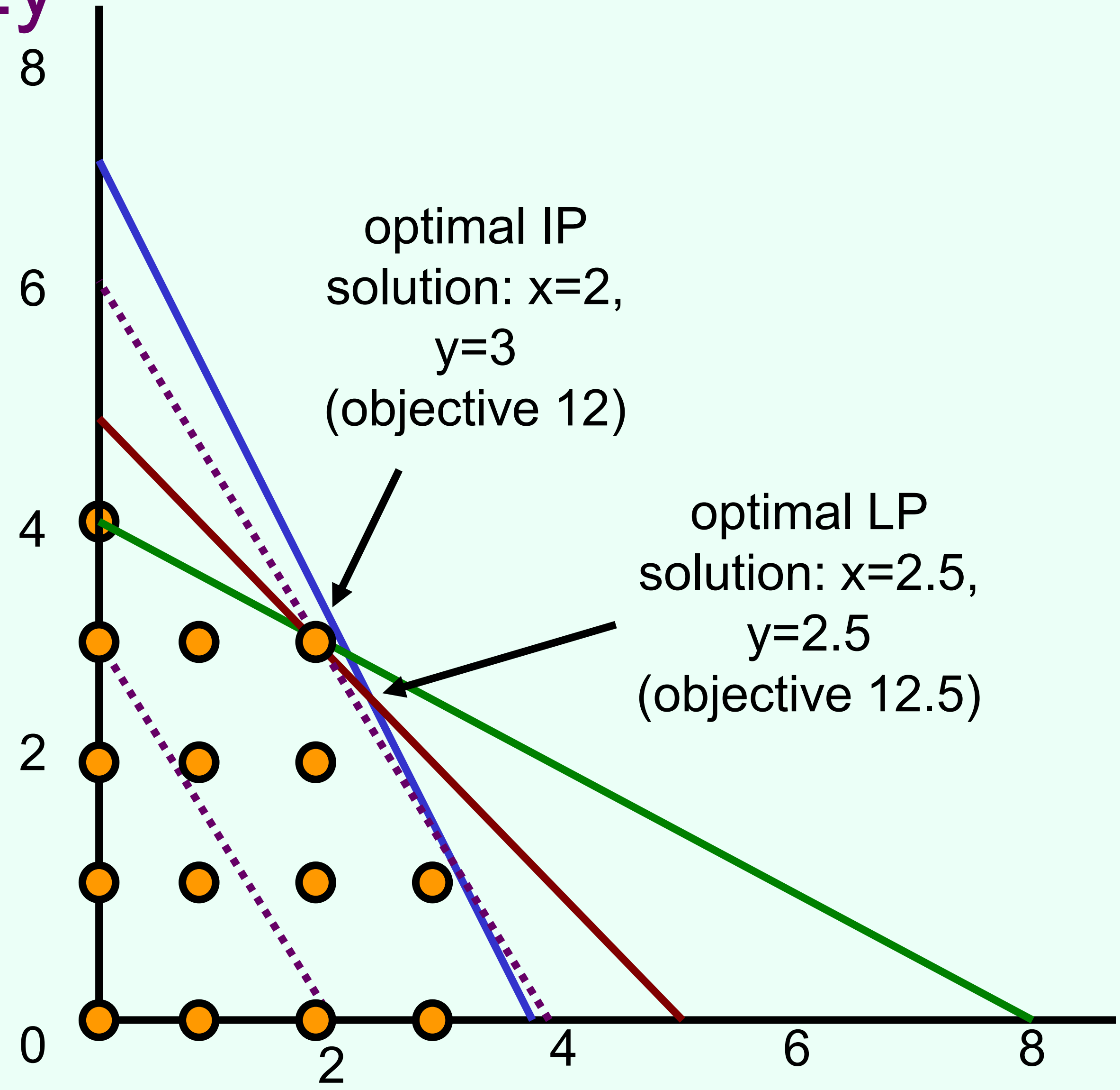
4x + 2y ≤ 15

x + 2y ≤ 8

x + y ≤ 5

x ≥ 0, integer

y ≥ 0, integer



Mixed integer (linear) program

maximize 3x + 2y

subject to

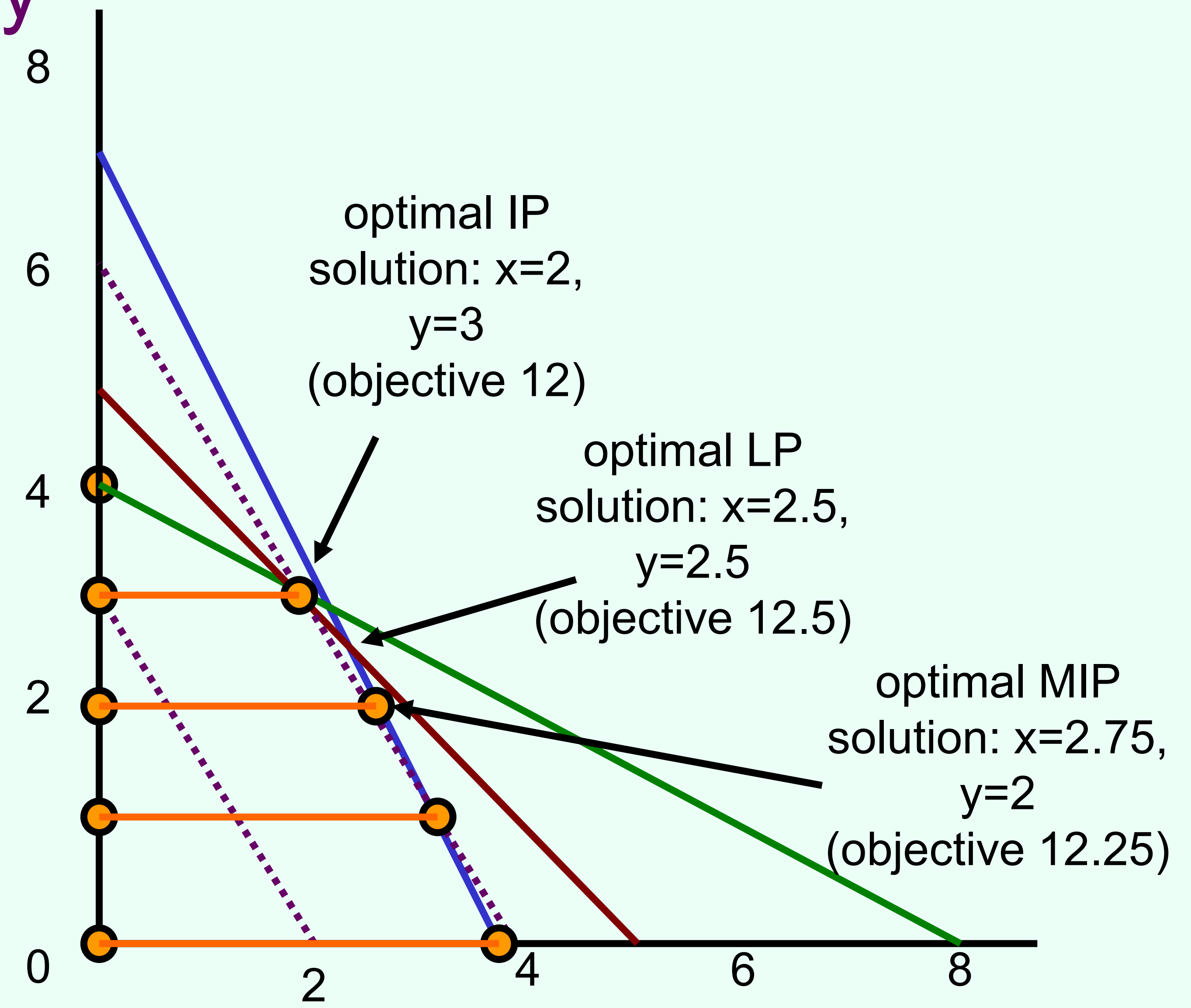
4x + 2y ≤ 15

x + 2y ≤ 8

x + y ≤ 5

x ≥ 0

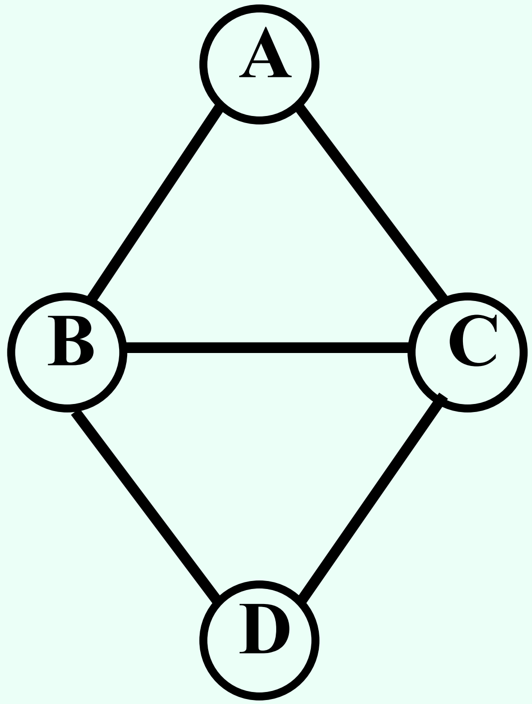
y ≥ 0, integer



Solving linear/integer programs

- Linear programs can be solved efficiently
  - Simplex, ellipsoid, interior point methods…
- (Mixed) integer programs are NP-hard to solve
  - Quite easy to model many standard NP-complete problems as integer programs (try it!)
  - Search type algorithms such as branch and bound
- Standard packages for solving these
  - GNU Linear Programming Kit, CPLEX, Gurobi, …
- LP relaxation of (M)IP: remove integrality constraints
  - Gives upper bound on MIP (~admissible heuristic)

Graph coloring as an integer program



- Let's say $x_{B,green}$ is 1 if B is colored green, 0 otherwise
- Must have $0 \leq x_{B,green} \leq 1$, $x_{B,green}$ integer
  - shorthand: $x_{B,green}$ in {0,1}
- Constraint that B and C can't both be green: $x_{B,green} + x_{C,green} \leq 1$
- Etc.
- Solving integer programs is at least as hard as graph coloring, hence NP-hard (we have reduced graph coloring to IP)

Satisfiability as an integer program

(x_1 OR x_2 OR NOT(x_4)) AND (NOT(x_2) OR NOT(x_3)) AND

…

becomes

for all x_j, $0 \le x_j \le 1$, x_j integer (shorthand: x_j in $\{0,1\}$)

$x_1 + x_2 + (1-x_4) \ge 1$

$(1-x_2) + (1-x_3) \ge 1$

…

Solving integer programs is at least as hard as satisfiability, hence NP-hard (we have reduced SAT to IP)

Try modeling other NP-hard problems as (M)IP!

Solving the integer program with DFS branch and bound

trick: for integer x and k, either $x \le k$ or $x \ge k+1$

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$

LP solution: x=2.5, y=2.5, obj = 12.5

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \ge 3$

LP solution: x=3, y=1.5, obj = 12

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \le 2$

LP solution: x=2, y=3, obj = 12

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \ge 3$
$y \ge 2$

LP solution: infeasible

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \ge 3$
$y \le 1$

LP solution: x=3.25, y=1, obj = 11.75

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \ge 3$
$y \le 1$
$x \ge 4$

LP solution: infeasible

maximize $3x + 2y$
subject to
$4x + 2y \le 15$
$x + 2y \le 8$
$x + y \le 5$
$x \ge 3$
$y \le 1$
$x \le 3$

LP solution: x=3, y=1, obj = 11

if LP solution is integral, we are done

Again with a more fortunate choice

maximize 3x + 2y
subject to
4x + 2y ≤ 15
x + 2y ≤ 8
x + y ≤ 5

LP solution: x=2.5, y=2.5, obj = 12.5

maximize 3x + 2y
subject to
4x + 2y ≤ 15
x + 2y ≤ 8
x + y ≤ 5
x ≥ 3

LP solution: x=3, y=1.5, obj = 12

maximize 3x + 2y
subject to
4x + 2y ≤ 15
x + 2y ≤ 8
x + y ≤ 5
x ≤ 2

LP solution: x=2, y=3, obj = 12

done!