Artificial Intelligence

Introduction to probability

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Uncertainty

- So far in course, everything deterministic
- If I walk with my umbrella, I **will not** get wet
- But: there is some chance my umbrella will break!
- Intelligent systems must take possibility of failure into account…
  - May want to have backup umbrella in city that is often windy and rainy
- … but should not be excessively conservative
  - Two umbrellas not worthwhile for city that is usually not windy
- Need quantitative notion of uncertainty

Probability

- Example: roll two dice
- Random variables:
  - X = value of die 1
  - Y = value of die 2
- Outcome is represented by an ordered pair of values (x, y)
  - E.g., (6, 1): X=6, Y=1
  - Atomic event or sample point tells us the **complete** state of the world, i.e., values of **all** random variables
- Exactly one atomic event will happen; each atomic event has a ≥0 probability; sum to 1
  - E.g., P(X=1 and Y=6) = 1/36

Y \ X	1	2	3	4	5	6
6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36

- An event is a proposition about the state (=subset of states)
  - X+Y = 7
- Probability of event = sum of probabilities of atomic events where event is true

Cards and combinatorics

- Draw a hand of 5 cards from a standard deck with 4*13 = 52 cards (4 suits, 13 ranks each)
- Each of the (52 choose 5) hands has same probability 1/(52 choose 5)
- Probability of event = number of hands in that event / (52 choose 5)
- What is the probability that…
  - no two cards have the same rank?
  - you have a flush (all cards the same suit?)
  - you have a straight (5 cards in order of rank, e.g., 8, 9, 10, J, Q)?
  - you have a straight flush?
  - you have a full house (three cards have the same rank and the two other cards have the same rank)?

Facts about probabilities of events

- If events A and B are disjoint, then
  - $P(A \text{ or } B) = P(A) + P(B)$
- More generally:
  - $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- If events A_1, …, A_n are disjoint and exhaustive (one of them must happen) then $P(A_1) + \ldots + P(A_n) = 1$
  - Special case: for any random variable, $\sum_x P(X=x) = 1$
- Marginalization: $P(X=x) = \sum_y P(X=x \text{ and } Y=y)$

Conditional probability

- We might know something about the world – e.g., "X+Y=6 or X+Y=7" – given this (and **only** this), what is the probability of Y=5?
- Part of the sample space is eliminated; probabilities are renormalized to sum to 1

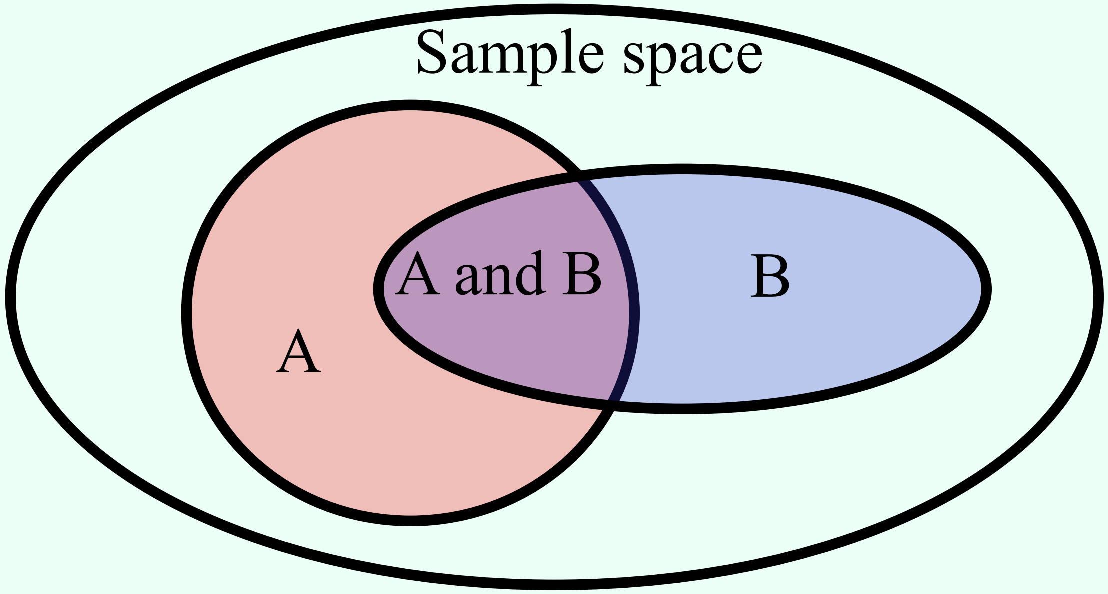
Y \ X	1	2	3	4	5	6
6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36

Y \ X	1	2	3	4	5	6
6	1/11	0	0	0	0	0
5	1/11	1/11	0	0	0	0
4	0	1/11	1/11	0	0	0
3	0	0	1/11	1/11	0	0
2	0	0	0	1/11	1/11	0
1	0	0	0	0	1/11	1/11

- P(Y=5 | (X+Y=6) or (X+Y=7)) = 2/11

Facts about conditional probability

- P(A | B) = P(A and B) / P(B)



- P(A | B)P(B) = P(A and B) = P(B | A)P(A)
- P(A | B) = P(B | A)P(A)/P(B)
  - Bayes' rule

Conditional probability and cards

- Given that your first two cards are Queens, what is the probability that you will get at least three Queens?
- Given that you have at least two Queens (not necessarily the first two), what is the probability that you have at least three Queens?
- Given that you have at least two Queens, what is the probability that you have three Kings?

How can we scale this?

- In principle, we now have a complete approach for reasoning under uncertainty:
  - Specify probability for every atomic event,
  - Can compute probabilities of events simply by summing probabilities of atomic events,
  - Conditional probabilities are specified in terms of probabilities of events: $P(A \mid B) = P(A \text{ and } B) / P(B)$
- If we have n variables that can each take k values, how many atomic events are there?

Independence

- Some variables have nothing to do with each other
- Dice: if X=6, it tells us nothing about Y
- P(Y=y | X=x) = P(Y=y)
- So: P(X=x and Y=y) = P(Y=y | X=x)P(X=x) = P(Y=y)P(X=x)
  - Usually just write P(X, Y) = P(X)P(Y)
  - Only need to specify 6+6=12 values instead of 6*6=36 values
  - Independence among 3 variables: P(X,Y,Z)=P(X)P(Y)P(Z), etc.
- Are the events "you get a flush" and "you get a straight" independent?

An example without cards or dice

	Rain in Beaufort	Sun in Beaufort
Rain in Durham	.2	.1
Sun in Durham	.2	.5

(disclaimer: no idea if these numbers are realistic)

- What is the probability of
  - Rain in Beaufort? Rain in Durham?
  - Rain in Beaufort, given rain in Durham?
  - Rain in Durham, given rain in Beaufort?
- Rain in Beaufort and rain in Durham are correlated

A possibly rigged casino

- With probability ½, the casino is rigged and has dice that come up *6* only 1/12 of the time, and *1* 3/12 of the time

Z=0 (fair casino)

Y \ X	1	2	3	4	5	6
6	1/72	1/72	1/72	1/72	1/72	1/72
5	1/72	1/72	1/72	1/72	1/72	1/72
4	1/72	1/72	1/72	1/72	1/72	1/72
3	1/72	1/72	1/72	1/72	1/72	1/72
2	1/72	1/72	1/72	1/72	1/72	1/72
1	1/72	1/72	1/72	1/72	1/72	1/72

Z=1 (rigged casino)

Y \ X	1	2	3	4	5	6
6	1/96	1/144	1/144	1/144	1/144	1/288
5	1/48	1/72	1/72	1/72	1/72	1/144
4	1/48	1/72	1/72	1/72	1/72	1/144
3	1/48	1/72	1/72	1/72	1/72	1/144
2	1/48	1/72	1/72	1/72	1/72	1/144
1	1/32	1/48	1/48	1/48	1/48	1/96

- What is P(Y=6)?
- What is P(Y=6|X=1)?
- Are they independent?

Conditional independence

- Intuition:
  - the only reason that X tells us something about Y,
  - is that X tells us something about Z,
  - and Z tells us something about Y
- If we already know Z, then X tells us nothing about Y
- $P(Y \mid Z, X) = P(Y \mid Z)$ or
- $P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)$
- "X and Y are conditionally independent given Z"

Medical diagnosis

- X: does patient have flu?
- Y: does patient have headache?
- Z: does patient have fever?
- $P(Y,Z|X) = P(Y|X)P(Z|X)$
- $P(X=1) = .2$
- $P(Y=1 \mid X=1) = .5$, $P(Y=1 \mid X=0) = .2$
- $P(Z=1 \mid X=1) = .4$, $P(Z=1 \mid X=0) = .1$
- What is $P(X=1|Y=1,Z=0)$?

(disclaimer: no idea if these numbers are realistic)

Conditioning can also **introduce** dependence

- X: is it raining?
  - $P(X=1) = .3$
- Y: are the sprinklers on?
  - $P(Y=1) = .4$
  - X and Y are independent
- Z: is the grass wet?
  - $P(Z=1 \mid X=0, Y=0) = .1$
  - $P(Z=1 \mid X=0, Y=1) = .8$
  - $P(Z=1 \mid X=1, Y=0) = .7$
  - $P(Z=1 \mid X=1, Y=1) = .9$

Not wet

	Raining	Not raining
Sprinklers	.012	.056
No sprinklers	.054	.378

Wet

	Raining	Not raining
Sprinklers	.108	.224
No sprinklers	.126	.042

- Conditional on Z=1, X and Y are **not** independent
- If you know Z=1, rain seems likely; then if you also find out Y=1, this “explains away” the wetness and rain seems less likely

Context-specific independence

- Recall P(X, Y | Z) = P(X | Z)P(Y | Z) really means: for all x, y, z,
- P(X=x, Y=y | Z=z) = P(X=x | Z=z)P(Y=y | Z=z)
- But it may not be true for *all z*
- P(Wet, RainingInLondon | CurrentLocation=New York) = P(Wet | CurrentLocation=New York)P(RainingInLondon | CurrentLocation=New York)
- But **not**
- P(Wet, RainingInLondon | CurrentLocation=London) = P(Wet | CurrentLocation=London)P(RainingInLondon | CurrentLocation=London)

Monty Hall problem

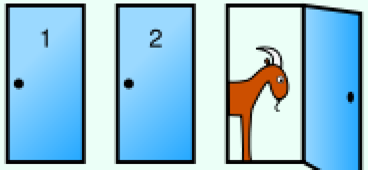
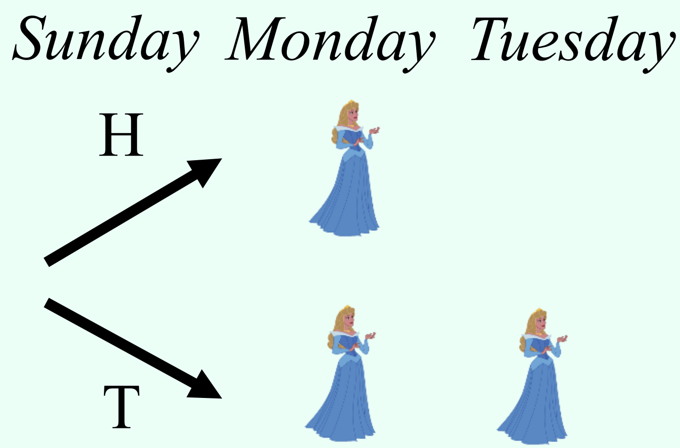


image taken from http://en.wikipedia.org/wiki/Monty_Hall_problem

- Game show participants can choose one of three doors
- One door has a car, two have a goat
  - Assumption: car is preferred to goat
- Participant chooses door, but not opened yet
- At least one of the other doors contains a goat; the (knowing) host will open one such door (flips coin to decide if both have goats)
- Participant is asked whether she wants to switch doors (to the other closed door) – should she?

Sleeping Beauty problem

- There is a participant in a study (call her Sleeping Beauty)
- On Sunday, she is given drugs to fall asleep
- A coin is tossed (H or T)
- If H, she is awoken on Monday, then made to sleep again
- If T, she is awoken Monday, made to sleep again, then again awoken on Tuesday



- Due to drugs she cannot remember what day it is or whether she has already been awoken once, but she remembers all the rules
- You're SB and you've just been awoken. What is your (subjective) probability that the coin came up H?

Expected value

- If Z takes numerical values, then the expected value of Z is $E(Z) = \sum_z P(Z=z)*z$
  - Weighted average (weighted by probability)
- Suppose Z is sum of two dice
- E(Z) = (1/36)*2 + (2/36)*3 + (3/36)*4 + (4/36)*5 + (5/36)*6 + (6/36)*7 + (5/36)*8 + (4/36)*9 + (3/36)*10 + (2/36)*11 + (1/36)*12 = 7
- Simpler way: E(X+Y)=E(X)+E(Y) (always!)
  - Linearity of expectation
- E(X) = E(Y) = 3.5

Linearity of expectation…

- If a is used to represent an atomic state, then $E(X) = \sum_x P(X=x)*x = \sum_x (\sum_{a:X(a)=x} P(a))*x = \sum_a P(a)*X(a)$
- $E(X+Y) = \sum_a P(a)*(X(a)+Y(a)) = \sum_a P(a)*X(a) + \sum_a P(a)*Y(a) = E(X)+E(Y)$

What is probability, anyway?

- Different philosophical positions:
- Frequentism: numbers only come from repeated experiments
  - As we flip a coin lots of times, we see experimentally that it comes out heads ½ the time
  - Problem: for most events in the world, there is no history of **exactly** that event happening
    - Probability that the Democratic candidate wins the next election?
- Objectivism: probabilities are a real part of the universe
  - Maybe true at level of quantum mechanics
  - Most of us agree that the result of a coin flip is (usually) determined by initial conditions + classical mechanics
- Subjectivism: probabilities merely reflect agents' beliefs