Artificial Intelligence

Introduction to probability

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Uncertainty

- So far in course, everything deterministic
- If I walk with my umbrella, I will not get wet
- But: there is some chance my umbrella will break!
- Intelligent systems must take possibility of failure into account...
  - May want to have backup umbrella in city that is often windy and rainy
- ... but should not be excessively conservative
  - Two umbrellas not worthwhile for city that is usually not windy

- Need **quantitative** notion of uncertainty
Probability

• Example: roll two dice
• Random variables:
  – $X =$ value of die 1
  – $Y =$ value of die 2
• Outcome is represented by an ordered pair of values $(x, y)$
  – E.g., $(6, 1): X=6, Y=1$
  – Atomic event or sample point tells us the complete state of the world, i.e., values of all random variables
• Exactly one atomic event will happen; each atomic event has a $\geq 0$ probability; sum to 1
  – E.g., $P(X=1 \text{ and } Y=6) = \frac{1}{36}$
• An event is a proposition about the state (=subset of states)
  – $X+Y = 7$
• Probability of event $=$ sum of probabilities of atomic events where event is true
Cards and combinatorics

• Draw a hand of 5 cards from a standard deck with $4 \times 13 = 52$ cards (4 suits, 13 ranks each)
• Each of the $(52 \text{ choose } 5)$ hands has same probability $1/(52 \text{ choose } 5)$
• Probability of event = number of hands in that event / $(52 \text{ choose } 5)$
• What is the probability that…
  – no two cards have the same rank?
  – you have a flush (all cards the same suit?)
  – you have a straight (5 cards in order of rank, e.g., 8, 9, 10, J, Q)?
  – you have a straight flush?
  – you have a full house (three cards have the same rank and the two other cards have the same rank)?
Facts about probabilities of events

- If events $A$ and $B$ are disjoint, then
  \[ P(A \text{ or } B) = P(A) + P(B) \]

- More generally:
  \[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

- If events $A_1, \ldots, A_n$ are disjoint and exhaustive (one of them must happen) then $P(A_1) + \ldots + P(A_n) = 1$
  
  – Special case: for any random variable, $\sum_x P(X=x) = 1$

- Marginalization: $P(X=x) = \sum_y P(X=x \text{ and } Y=y)$
Conditional probability

- We might know something about the world – e.g., “X+Y=6 or X+Y=7” – given this (and only this), what is the probability of Y=5?
- Part of the sample space is eliminated; probabilities are renormalized to sum to 1

\[
P(Y=5 \mid (X+Y=6) \text{ or } (X+Y=7)) = \frac{2}{11}
\]
Facts about conditional probability

- $P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$

- $P(A \mid B)P(B) = P(A \text{ and } B) = P(B \mid A)P(A)$

- $P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$

  – Bayes’ rule
Conditional probability and cards

• Given that your first two cards are Queens, what is the probability that you will get at least three Queens?

• Given that you have at least two Queens (not necessarily the first two), what is the probability that you have at least three Queens?

• Given that you have at least two Queens, what is the probability that you have three Kings?
How can we scale this?

• In principle, we now have a complete approach for reasoning under uncertainty:
  – Specify probability for every atomic event,
  – Can compute probabilities of events simply by summing probabilities of atomic events,
  – Conditional probabilities are specified in terms of probabilities of events: \( P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \)

• If we have \( n \) variables that can each take \( k \) values, how many atomic events are there?
Independence

• Some variables have nothing to do with each other
• Dice: if X=6, it tells us nothing about Y
• \( P(Y=y \mid X=x) = P(Y=y) \)
• So: \( P(X=x \text{ and } Y=y) = P(Y=y \mid X=x)P(X=x) = P(Y=y)P(X=x) \)
  – Usually just write \( P(X, Y) = P(X)P(Y) \)
  – Only need to specify 6+6=12 values instead of 6*6=36 values
  – Independence among 3 variables: \( P(X,Y,Z)=P(X)P(Y)P(Z), \text{ etc.} \)
• Are the events “you get a flush” and “you get a straight” independent?
An example without cards or dice

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(disclaimer: no idea if these numbers are realistic)

• What is the probability of
  – Rain in Beaufort? Rain in Durham?
  – Rain in Beaufort, given rain in Durham?
  – Rain in Durham, given rain in Beaufort?

• Rain in Beaufort and rain in Durham are correlated
A possibly rigged casino

- With probability $\frac{1}{2}$, the casino is rigged and has dice that come up 6 only $\frac{1}{12}$ of the time, and $\frac{13}{12}$ of the time.

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- What is $P(Y=6)$?
- What is $P(Y=6|X=1)$?
- Are they independent?
Conditional independence

• Intuition:
  – the only reason that \( X \) tells us something about \( Y \),
  – is that \( X \) tells us something about \( Z \),
  – and \( Z \) tells us something about \( Y \)
• If we already know \( Z \), then \( X \) tells us nothing about \( Y \)
• \( P(Y | Z, X) = P(Y | Z) \) or
• \( P(X, Y | Z) = P(X | Z)P(Y | Z) \)
• “\( X \) and \( Y \) are conditionally independent given \( Z \)”
Medical diagnosis

- X: does patient have flu?
- Y: does patient have headache?
- Z: does patient have fever?

- $P(Y,Z|X) = P(Y|X)P(Z|X)$
- $P(X=1) = .2$
- $P(Y=1 \mid X=1) = .5$, $P(Y=1 \mid X=0) = .2$
- $P(Z=1 \mid X=1) = .4$, $P(Z=1 \mid X=0) = .1$
- What is $P(X=1|Y=1,Z=0)$?

(disclaimer: no idea if these numbers are realistic)
Conditioning can also introduce dependence

- **X**: is it raining?
  - \( P(X=1) = .3 \)

- **Y**: are the sprinklers on?
  - \( P(Y=1) = .4 \)
  - \( X \) and \( Y \) are independent

- **Z**: is the grass wet?
  - \( P(Z=1 \mid X=0, Y=0) = .1 \)
  - \( P(Z=1 \mid X=0, Y=1) = .8 \)
  - \( P(Z=1 \mid X=1, Y=0) = .7 \)
  - \( P(Z=1 \mid X=1, Y=1) = .9 \)

- Conditional on \( Z=1 \), \( X \) and \( Y \) are **not** independent

- If you know \( Z=1 \), rain seems likely; then if you also find out \( Y=1 \), this “explains away” the wetness and rain seems less likely.
**Context-specific independence**

- Recall $P(X, Y | Z) = P(X | Z)P(Y | Z)$ really means: for all $x, y, z$,
  
  $$P(X=x, Y=y | Z=z) = P(X=x | Z=z)P(Y=y | Z=z)$$

- But it may not be true for *all* $z$

  - $P(Wet, RainingInLondon | CurrentLocation=New York) = P(Wet | CurrentLocation=New York)P(RainingInLondon | CurrentLocation=New York)$

  - But *not*

Monty Hall problem

- Game show participants can choose one of three doors
- One door has a car, two have a goat
  - Assumption: car is preferred to goat
- Participant chooses door, but not opened yet
- At least one of the other doors contains a goat; the (knowing) host will open one such door (flips coin to decide if both have goats)
- Participant is asked whether she wants to switch doors (to the other closed door) – should she?
Sleeping Beauty problem

- There is a participant in a study (call her Sleeping Beauty)
- On Sunday, she is given drugs to fall asleep
- A coin is tossed (H or T)
- If H, she is awoken on Monday, then made to sleep again
- If T, she is awoken Monday, made to sleep again, then again awoken on Tuesday

Due to drugs she cannot remember what day it is or whether she has already been awoken once, but she remembers all the rules

You’re SB and you’ve just been awoken. What is your (subjective) probability that the coin came up H?
Expected value

- If Z takes numerical values, then the expected value of Z is \( E(Z) = \sum_z P(Z=z) \cdot z \)
  - Weighted average (weighted by probability)

- Suppose Z is sum of two dice
  \[
  E(Z) = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12 = 7
  \]

- Simpler way: \( E(X+Y) = E(X) + E(Y) \) (always!)
  - Linearity of expectation

- \( E(X) = E(Y) = 3.5 \)
Linearity of expectation…

• If $a$ is used to represent an atomic state, then $E(X) = \sum_x P(X=x) \cdot x = \sum_x \left( \sum_{a: X(a) = x} P(a) \right) \cdot x = \sum_a P(a) \cdot X(a)$

• $E(X+Y) = \sum_a P(a) \cdot (X(a)+Y(a)) = \sum_a P(a) \cdot X(a) + \sum_a P(a) \cdot Y(a) = E(X)+E(Y)$
What is probability, anyway?

- Different philosophical positions:
  - **Frequentism**: numbers only come from repeated experiments
    - As we flip a coin lots of times, we see experimentally that it comes out heads $\frac{1}{2}$ the time
    - Problem: for most events in the world, there is no history of exactly that event happening
      - Probability that the Democratic candidate wins the next election?
  - **Objectivism**: probabilities are a real part of the universe
    - Maybe true at level of quantum mechanics
    - Most of us agree that the result of a coin flip is (usually) determined by initial conditions + classical mechanics
  - **Subjectivism**: probabilities merely reflect agents’ beliefs