Batch Approximate Policy Iteration
(Linear Models, LSTD, and LSPI)

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LSPI is joint work with Michail Lagoudakis
Equivalence between the linear model and LSTD is joint work
with Li, Littman, Painter-Wakefield and Taylor

Online versus Batch RL

• **Online RL:**
  – Concurrent data collection and optimization
  – Example: TD-learning and Q-learning

• **Batch RL:**
  – Decouples data collection and optimization
  – Generate database of experiences in the environment
  – Use the fixed set of experiences to optimize/learn a policy
  – Example: Fitted Value Iteration

• **Online vs. Batch:**
  – Batch algorithms are often more “data efficient” and stable
  – Batch algorithms may ignore the exploration-exploitation problem, and do their best with the data they have (e.g. off-policy data from another source)
  – Not necessarily a crisp distinction: Batch algorithms can be interleaved with exploration to augment database
Linear Model Approximation
(Policy Evaluation Case)

- Linearly independent features \( \Phi = (\phi_1, ..., \phi_k) \) \( (n \times k) \)
- Want \( R_\Phi = \text{reward model} \) \( (k \times 1) \) w/smallest \( L_2 \) error:
  \[
  \Phi R_\Phi \approx R
  \]
  \[
  R_\Phi = (\Phi^T \Phi)^{-1} \Phi^T R = \Pi_\Phi R
  \]
- Want \( P_\Phi = \text{feature } \times \text{feature model} \) \( (k \times k) \) w/ smallest \( L_2 \) error
  \[
  \Phi P_\Phi \approx \Pi_\Phi (\approx \Phi') \quad \text{Expected (or sampled)}
  \]
  \[
  P_\Phi = \Pi_\Phi \Pi_\Phi (\approx \Pi_\Phi \Phi') \quad \text{next feature values} \( (n \times k) \)
  \]

State Values Under Linear Model

- Bellman equation:
  \[
  V(s) = R(s) + \gamma \sum_s P(s \mid s) V(s)
  \]
- Bellman equation in terms of features
  \[
  V(\phi(s)) = R(\phi(s)) + \gamma V(\phi(s))
  \]
  \[
  = \phi(s)^T R_\phi + \gamma V(\phi(s)^T P_\phi)
  \]
  \[
  = \phi(s)^T R_\phi + \gamma (\phi(s)^T P_\phi R_\phi + \gamma V(\phi(s)^T P_\phi))
  \]
  \[
  = \sum_{\tau=0}^\infty \gamma^\tau \phi(s)^T P_\phi R_\phi = \phi(s)^T \left( \sum_{\tau=0}^\infty \gamma^\tau P_\phi \right) R_\phi = \phi(s)^T (I - \gamma P_\phi)^{-1} R_\phi
  \]
Value Function of the Linear Model

- Value function is in span(\(\Phi\))
- Can express value functions as \(\Phi w\)
- If \(V\) is bounded, then:

\[
\begin{align*}
    w &= (I - \gamma P_\phi)^{-1} R_\phi \\
    &= (I - \gamma (\Phi^T \Phi)^{-1} \Phi^T P \Phi)^{-1} (\Phi^T \Phi)^{-1} \Phi^T R \\
    &\approx (I - \gamma (\Phi^T \Phi)^{-1} \Phi^T \Phi')^{-1} (\Phi^T \Phi)^{-1} \Phi^T R
\end{align*}
\]

- Note similarity to conventional solution:

\[
V^* = (I - \gamma P)^{-1} R
\]

Solving for the Value Function Directly

- LSTD \([\text{Bradtke & Barto}]\) aims to estimate the value function directly from samples
- Give \(\Phi, \Phi', \) and sampled \(R\), LSTD computes

\[
\begin{align*}
    w &= (\Phi^T \Phi - \gamma \Phi^T P \Phi)^{-1} \Phi^T R
\end{align*}
\]

- Where does this come from?
Linear Fixed Point

- $\Pi_\phi V =$ weights of projection of $V$ into $\text{span}(\Phi)$

$LSTD$ solves for the linear fixed point:

$$w_\phi = \Pi_\phi \Gamma \phi w_\phi$$

Deriving LSTD

- Recall fixed-point equation for policies

$$V(s) = R(s, \pi(s)) + \gamma \sum_{s'} P(s|s) V(s')$$

- For linear value function approximation:

$$V = R + \gamma \sum_s P(s|s) \phi w$$

- Problem: This might not be in $\text{span}(\Phi)$

- Solution project back into $\text{span}(\Phi)$

$$\hat{V} = \phi w = \prod_\phi (R + \gamma P \phi w) = \prod_\phi (R + \gamma \Phi' w)$$

- Substituting least squares projection into this gives:

$$\phi w = \phi (\phi^T \phi)^{-1} \phi^T (R + \gamma P \phi w)$$

$$w = (\phi^T \phi - \gamma \phi^T P \phi)^{-1} \phi^T R = (\phi^T \phi - \gamma \phi^T \phi')^{-1} \phi^T R$$
LSTD Solution = Linear Model Solution

- LSTD solution: \( w = (\Phi^T \Phi - \gamma \Phi^T \Phi')^{-1} \Phi^T R \)

- Linear model solution:
  \[ w = \left( I - \gamma (\Phi^T \Phi)^{-1} \Phi^T \Phi' \right)^{-1} (\Phi^T \Phi)^{-1} \Phi^T R \]
  \[ = (\Phi^T \Phi - \gamma \Phi^T \Phi')^{-1} \Phi^T R \]

- Conclusion: LSTD and solution to linear model approximation are identical

- Historical notes:
  - Indicator function case was clear from the beginning, but
  - People did not initially realize this equivalence in full generality

Approximate Policy Iteration with LSTD

**Policy Iteration:** iterates between policy improvement and policy evaluation

**Idea:** use LSTD for approximate policy evaluation in PI

Start with random weights \( w \) (i.e. value function)

Repeat Until Convergence

\( \pi(s) = \text{greedy}(\Phi w) \)

Evaluate \( \pi \) using LSTD

- Generate sample trajectories of \( P\pi \)
- Use LSTD to produce new weights \( w \)
  \( (w \) gives an approx. value function of \( \pi \) \)
What Breaks?

• No way to execute greedy policy without a model

• Approximation is biased by current policy
  – We only approximate values of states we see when executing the current policy
  – LSTD is a *weighted* approximation toward those states

• Can result in Learn-forget cycle of policy iteration
  – Drive off the road; learn that it’s bad
  – New policy never does this; forgets that it’s bad

LSPI

• LSPI replaces LSTD with a new algorithm: LSTDQ
• LSTD: produces a value function
  – Requires sample from policy under consideration
• LSTDQ: produces a Q-function \( Q_\pi \), not necessarily \( Q^* \)
  – Can learn Q-function for policy from any (reasonable) set of samples---sometimes called an off-policy method
  – No need to collect samples from current policy
• Disconnects policy evaluation from data collection
  – Permits reuse of data across iterations
Computing Q-functions w/LSTDQ

- Suppose we have samples of form \((s,a,r,s')\)
- Expand our state space to include the actions taken as part of the state
- View our samples as \(([s,a],r,[s',\pi(s')])\)
  - Q: Is this valid if we didn’t really take \(\pi(s')\) in \(s'\)?
  - A: Yes, because the right hand side of the Bellman equation just uses our linear approximation:

  \[
  Q(s,a) = R(s) + \sum_s P(s|s,a)\mathbb{V}(s') = R(s) + \sum_s P(s|s,a)Q(s',\pi(s)) \\
  = R(s) + \sum_s P(s|s,a)\mathbb{\phi}_{\pi(s)}(s)w_{\pi(s)}
  \]

Implementing LSPI

- Implement LSTDQ as LSTD with expanded feature set:
  - For \(A\) actions, make \(A\) copies of our features
  - Features are replaced with product of features and indicators
  - \(\phi(s)\rightarrow\phi_{a1}(s)=\phi(s)(a_1), \phi_{a2}(s)=\phi(s)(a_2),...\)
- Each time we run LSTDQ, our Q values encode the next policy, i.e.

  \[\pi_{i+1}(s) = \text{arg max}_a Q_i(s,a)\]
Running LSPI

1. Collect a database of \((s,a,r,s')\) experiences
2. Start \(w/\)random weights (= random policy)
3. Repeat until convergence*
   - Evaluate current policy against database
     • Run LSTDQ to generate new set of weights
     • New weights imply new Q-function and hence new policy
   - Replace current weights with new weights

*Nota bene: LSPI may not converge; need some notion of “convergence”

What’s under the hood?

• At each iteration, we’re basically doing LSTD:

\[
W = (\Phi^T\Phi - \gamma\Phi^T\Phi')^{-1}\Phi^TR
\]

• What changes at each iteration?

\[
W = (\Phi^T\Phi - \gamma\Phi^T\Phi')^{-1}\Phi^TR
\]

When the policy changes, the \(\pi(s')\) part of \([s',\pi(s')]\), changes. That’s it!
In practice, all that changes is which indicators are active in \(\Phi'\).
Everything else can be cached.
Example Results: Bicycle Riding

- Randlov and Alstrom simulator
- Watch random controller operate bike
- Collect ~40,000 (s,a,r,s’) samples
- Pick 20 simple basis functions (×5 actions)
- Make 5-10 passes over data (PI steps)

Result:
Controller that balances and rides to goal
LSPI Robustness

![Graph showing robustness over episodes]

Good Things About LSPI

- Possibly the first fully off-policy approximate policy iteration algorithm with data collection and policy iteration fully decoupled
- Easy to implement
- Fast if you have good linear algebra libraries
- Enabled a clear focus on function approximation issues by decoupling policy solving and data collection
So, what’s the bad news?

- $(k (#A))^2$ can sometimes be big
  - Lots of storage
  - Matrix inversion can be expensive
- Only works with linear approximation – requires careful feature engineering for hard(er) problems
- While avoidance of exploration/data collection issues is a feature, it’s also a bug since it does not provide a solution for these challenges

LSPI as an Instance of Approximate Policy Iteration

- Analysis of approximate policy iteration from **Neuro-Dynamic Programming**, Bertsekas and Tsisiklis, 1996

\[
\lim_{k \to \infty} \sup \|V^{\pi_k} - V^*\|_\infty \leq \frac{\delta + 2\gamma \epsilon}{(1 - \gamma)^2}
\]

- $\epsilon = \text{worst max-norm value function error}$
  \[
  \epsilon = \max_k \|V^{\pi_k} - \hat{V}^{\pi_k}\|_\infty
  \]
- $\delta = \text{max-norm policy update error}$
  \[
  \delta = \max_k \|TV^{\pi_k} - \hat{T}V^{\pi_k}\|_\infty
  \]
LSPI Error Bounds in Practice

- No guarantee matrix is invertible or well-conditioned – some choices of features can lead to ill-conditioned matrices and weird/unstable Q-values

- No practical guarantee on $\varepsilon$

- $\delta = 0$

Fitted Q-Iteration

- Can we apply the Q-function trick to fitted value iteration?
- Yes!
- Fitted Q-iteration algorithm:
  - Randomly initialize approximate Q function $Q_0$
  - $i=0$
  - Repeat until done*
    - Sample states $s^1...s^m$
    - $\pi = \text{greedy}(Q_i)$
    - Fit $Q_{i+1}^\pi$ on $T^\pi Q_i(s^1)...T^\pi Q_i(s^m)$
    - $i=i+1$

- Achieves value iteration w/o a model
LSPI vs. Fitted Q-iteration

<table>
<thead>
<tr>
<th></th>
<th>Representation Expressiveness</th>
<th>Can Diverge</th>
<th>Can Oscillate</th>
<th>Reuse Training Data Across Policies</th>
<th>Underlying MDP Solution Method</th>
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<tr>
<td>LSPI</td>
<td>linear</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Policy Iteration</td>
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<td>Arbitrary</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Value Iteration</td>
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Question: Is there something in between these, like modified approximate policy iteration?

Answer: Frozen target network in DQN is a step in this direction.

Final Thoughts

- By using Q-functions, batch methods can do policy improvement w/o a model
- Does not explicitly address data collection (exploration) – both a pro and a con
- In practice, performance depends heavily upon:
  - Choice of features (both LSPI and fitted Q)
  - Choice of approximation architecture (fitted Q)
  - Distribution of training samples – how exploration is taken into account