Introduction to Approximation Algorithms

Ron Parr
CPS 570

Covered Today

- Approximation in general
- Set cover
- A greedy algorithm for set cover
- Submodularity
- Generic, greedy algorithm exploiting submodularity

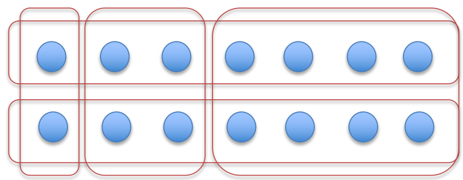
Some Intractable Combinatorial Optimization Problems

- Find the lowest cost traveling salesman tour
- Color a graph with the fewest possible colors
- Cover with the lowest number of vertices/sets

Set Cover

- Input:
  - A set of atoms: $A=a_1 \ldots a_n$
  - A set of sets: $S=s_1 \ldots s_m$
  - Each set contains 1 or more atoms
- Optimization question: Can you choose k elements from S such that every element of A is in at least one of these k? (This is a called a cover.)
- Decision question: Exist a cover of size k or less?
- NP-hard

Set Cover Example



14 atoms
5 sets

Hardness of Set Cover

- Karp showed that set cover is NP-complete (classic paper on reading list)
- Satisfiability reduces to clique
- Clique reduces to node (vertex) cover
- Node cover reduces to set cover

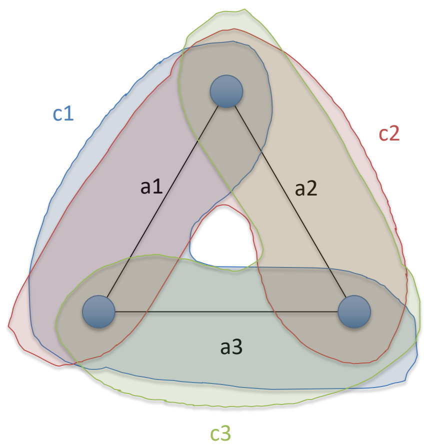
Node (vertex) Cover

- Input:
  - Graph G=E,V
- Optimization question: What is the smallest set of vertices such that every edge is incident upon one of the vertices
- Decision question: Does there exist a set of vertices of size k such that every edge is incident on at least one vertex in k

Reduce Node Cover to Set Cover

- Remember: Must solve node cover w/set cover
- For each edge in the node cover problem, we create an atom in the set cover problem
- For each node in the node cover problem, we create a set s.t. elements of the set correspond to edges incident to the node
- Observe that a set cover of size k exists iff a node cover of size k exists

Reduction Example



- Edges in node cover -> atoms (a1…a3) in set cover
- Each node in node cover -> set with all adjacent edges in set cover (c1..c3)

Real Problems Abstracted by Set Cover

- Sensor placement:
  - You have sensors to place in m different locations
  - Each location can observe some fraction of your n targets
  - Find the most efficient sensor allocation to see all targets
- Buying bundles of goods
  - Different vendors offer package deals on different combinations of products (flat rate shipping)
  - Buy all the products you need in the smallest number of transactions
- Choosing advertising outlets
  - Different stations/newspapers/sites cover different, possibly overlapping markets
  - Try to cover markets with smallest number of ads
- Choosing test cases for code/hardware/exams
  - Different tests exercise different (overlapping) parts of the system
  - Try to verify system in smallest number of tests

Why use approximation?

- Many problems we want to solve are NP-hard optimization problems w/associated NP-complete decision problems

- Different notions of approximation
  - Search for a “pretty good” answer
  - Return an optimal answer in some cases (fail in others?)
  - Return an answer that is an additive factor from optimal: result = optimal +/- ε
  - Return an answer that a multiplicative factor from optimal: result/approximation = ε
  - For a given resource level, achieve a lower performance value?
  - For a given performance level, consume more resources?

So, what do we do in, e.g., set cover?

- Settle for a larger k (more sets)?
  - What if we don’t need the absolute smallest k?
  - Is there an algorithm that gives something close to the smallest but still covers everything?

- Settle for less than full coverage
  - What if we have only k resources?
  - Is there an algorithm that gives us something close to the best coverage using only k?

Greedy Algorithms

- Greedy algorithms are a general class of algorithms that, *loosely speaking*, make a choice that gives maximal short term improvement, without considering subsequent choices

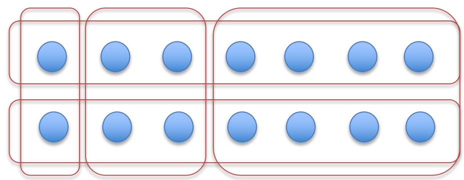
- Examples of greedy behavior:
  - Picking the class that is most interesting to you first (ignoring that this might cause scheduling problems with other classes)
  - Positioning a sensor so that it sees the highest number of targets (while ignoring subsequent choices)

Greedy Set Cover

- Repeat until done*
  - For each set not added, check how many previously uncovered atoms it would add
  - Add the set with the biggest increase in the number of atoms covered

- *What is "done"
  - Max of k elements added, ***or***
  - All elements covered

What does greedy do here?



What price greed?

- Assume we have a budget of k
- Optimal picks: $O_1 \ldots O_k$, covering n atoms
- Greedy picks $G_1 \ldots G_k$, covering x atoms
- What is the relationship between x and n?

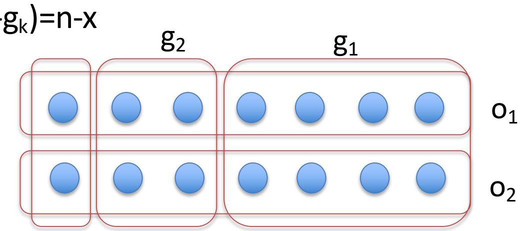
What price greed (2)?

- o_i = number of *new* elements covered by O_i
- g_i = number of *new* elements covered by G_i
- "new" means not previously covered by 1…i-1

- $n = o_1+o_2+\ldots+o_k$
- $x = g_1+g_2+\ldots+g_k$

What price greed (3)?

- Suppose $o_i>g_i$, $i>1$
- Q: Why didn't greedy pick o_i?
- A: The only reason would be if greedy already covered o_i-g_i of the elements in o_i in some g_j, $j<i$
- $x \geq (o_1-g_1)+(o_2-g_2)+\ldots+(o_k-g_k)=n-x$
- $2x \geq n$
- $x \geq n/2$



- Conclusion: For fixed k, greedy gets a least half as much coverage as optimal

What about minimizing k to achieve full coverage?

- Suppose optimal coverage uses k sets to cover n atoms
- We run greedy until it covers everything, taking $h \geq k$ sets
- Analyze greedy's h choices in batches of k
  - Greedy covers at least n/2 in first batch of k
  - Second batch of k covers at least half of remaining atoms. Why? Same analysis can be repeated.
- Conclusion: greedy requires at $O(k\log_2 n)$ sets
- Note: Our bounds are **not tight** in this case. Better proof exploiting submodularity is possible.

Applying to Other Problems

- If we have a good approximation scheme for one NP-hard problem, does this imply a good approximation scheme for others? (e.g. transform to set cover, then approximate the transformed problem)
- Depends upon what what you mean by *"good"*…
- The polynomial factor can be a killer here
- Conclusion: Approximation algorithms will tend to be problem specific unless one discovers a more general approach to approximation

Submodularity

- f is a function defined on sets
- Monotone if:

$$X \subseteq Y : f(Y) \geq f(X)$$

- Submodular if

$$X, Y \subseteq \Omega, X \subseteq Y, z \notin Y : f(X \cup \{z\}) - f(X) \geq f(Y \cup \{z\}) - f(Y)$$

In English

- Monotonicity: Bigger is better
  (though not strictly)

- Submodularity:
  – Adding to a subset has at least as much “bang” as adding to a superset, or
  – Diminishing returns for adding to bigger sets

Set Cover?

- Does set cover fit this framework?
- f = number of atoms covered

- Is it monotone?
- Is it submodular?

Maximizing Monotone Submodular Set Functions

- Goal: Given budget of k sets, maximize f
- This is NP-hard in general ☹

- Greedy algorithm for maximizing f that is a
  - Non-negative
  - Monotone
  - Submodular

  set function is a 1-1/e (~0.63) factor from optimal for budget k
- Similar argument to set cover to gives a resource bound
- Proof in reading, similar to our 2X bound, but a little more subtle

- Provides a generic procedure for analyzing greedy algorithms for certain classes of hard problems ☺

Greedy Submodular Maximization

- Input: set of sets Ω, score function f
- X = {}
- Repeat until "done"
  - Find set ω in Ω that maximizes $f(X+\omega)$
  - Add ω to X
- "done":
  - $|X| = k$, or
  - $f(X)$ = some target value

Greedy Set Cover and Submodularity

- Our greedy algorithm for set cover can be understood as an instance of the greedy approach for submodular set functions
- Conclusion: We get a tighter bound for free!
- $(1-1/e > ½)$

Exploiting Submodularity

- Frequently used to justify greedy approaches that otherwise would have had computation/implementation ease as their only justification
- Impactful in, e.g., sensor network community

Conclusions

- Avoid worst consequences NP-hardness with clever approximation algorithms (or clever analysis of simple algorithms)
- Caveats:
  - Not all problems admit good approximate solutions
  - Specific approximation techniques for one problem don't necessarily apply to others
- Some generic approaches exist:
  - Greedy algorithms sometimes do well
  - Submodularity provides a generic framework for analyzing certain types of greedy algorithms
  - Other families of approaches exist as well – rounding, LP relaxations, etc.