Chapter 7.2

Theorem Given NPDA M that accepts by final state, ∃ NPDA M' that accepts by empty stack s.t. L(M)=L(M').

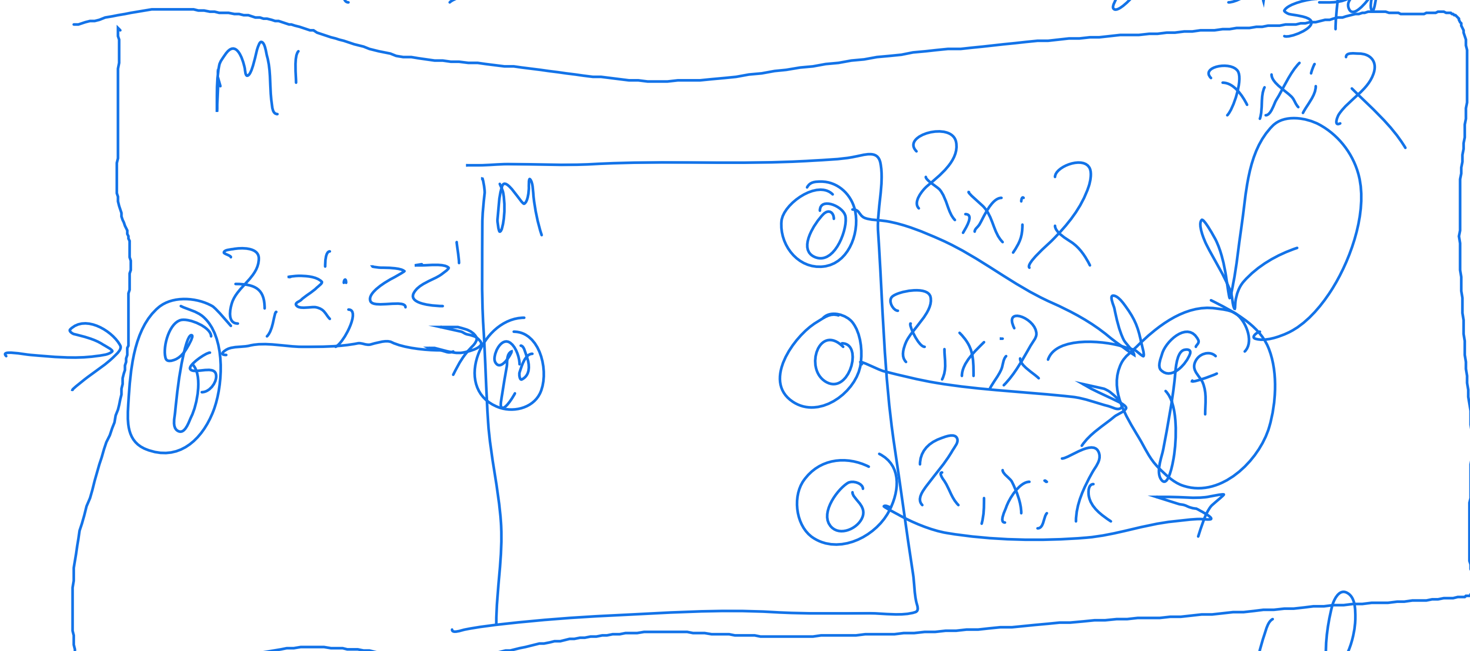
- **Proof (sketch)**

  M=(Q,Σ**,**Γ**,**δ**,**q_0**,z,F)**

  Construct M'=(Q',Σ**,**Γ'**,**δ'**,**q_s**,z',F')**

$Q' = Q \cup \{q_s, q_f\}$ $F' = \{\}$

$\Gamma' = \Gamma \cup \{z'\}$ note $z' \notin \Gamma$ q_s new start state



where X is any symbol in Γ'

M' only empty in q_f

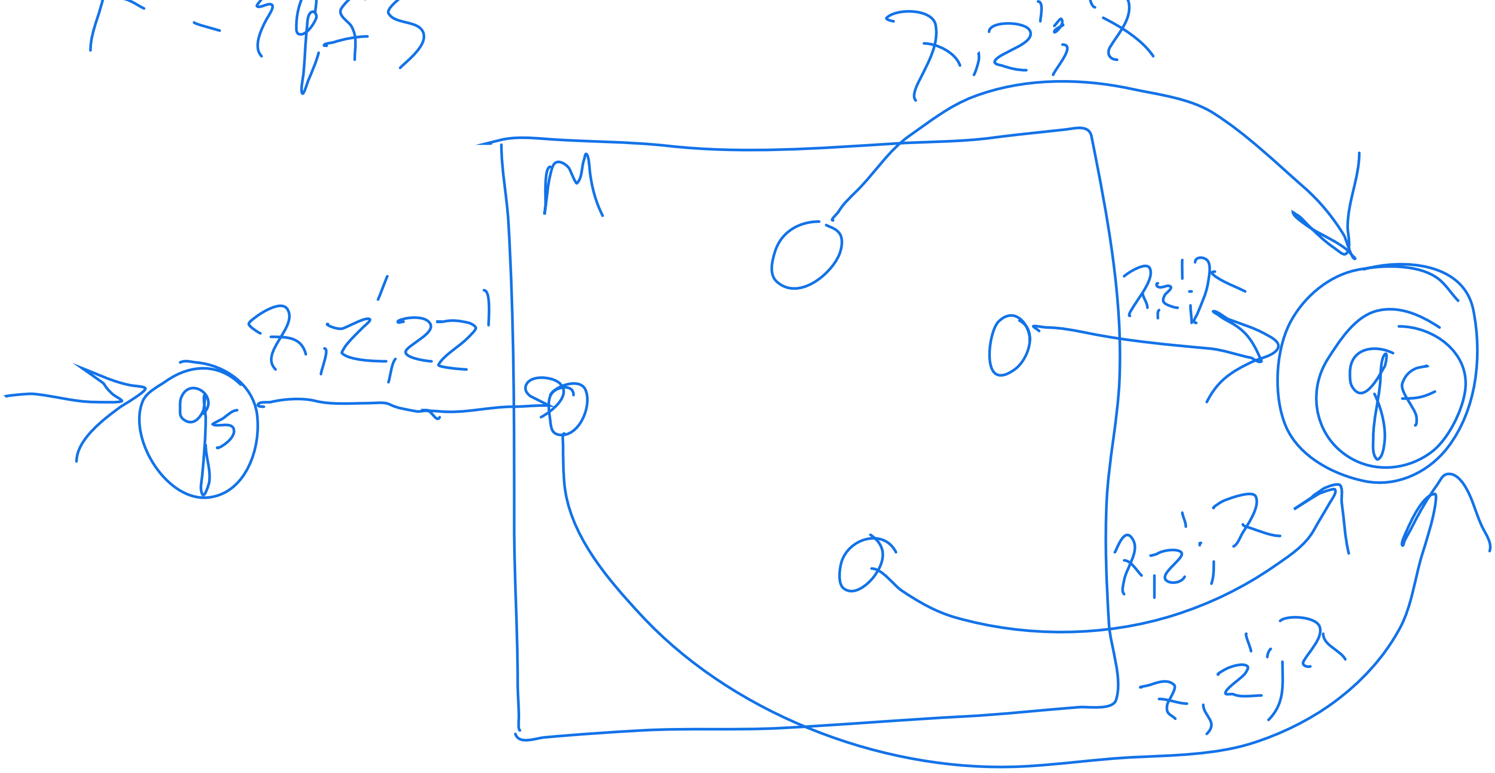
Theorem Given NPDA M that accepts by empty stack, ∃ NPDA M' that accepts by final state.

- **Proof: (sketch)**
  $\mathbf{M}=(\mathbf{Q},\Sigma,\Gamma,\delta,q_0,\mathbf{z},\mathbf{F})$
  Construct $\mathbf{M'}=(\mathbf{Q'},\Sigma,\Gamma',\delta',q_s,\mathbf{z'},\mathbf{F'})$

$Q' = Q \cup \{q_s, q_f\}$, $\Gamma' = \Gamma \cup \{z'\}$

$F' = \{q_f\}$

q_s new start state



$(q_f, \lambda) \in \delta(q, \lambda, z')$ for all $q \in Q$

Theorem For any CFL L not containing λ, $\exists$ an NPDA M s.t. L=L(M).

- **Proof (sketch)**

  Given (λ-free) CFL L.

  $\Rightarrow \exists$ CFG G such that L=L(G).

  $\Rightarrow \exists$ G' in GNF, s.t. L(G)=L(G').

  G'=(V,T,S,P). All productions in P are of the form:

$$A \to ax$$

$$A \in V,\ a \in T,\ x \in V^*$$

Construct NPDA M

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

$Q = \{q_0, q_1, q_f\}$, $\Sigma = T$, $\Gamma = V \cup \{z\}$

$F = \{q_f\}$

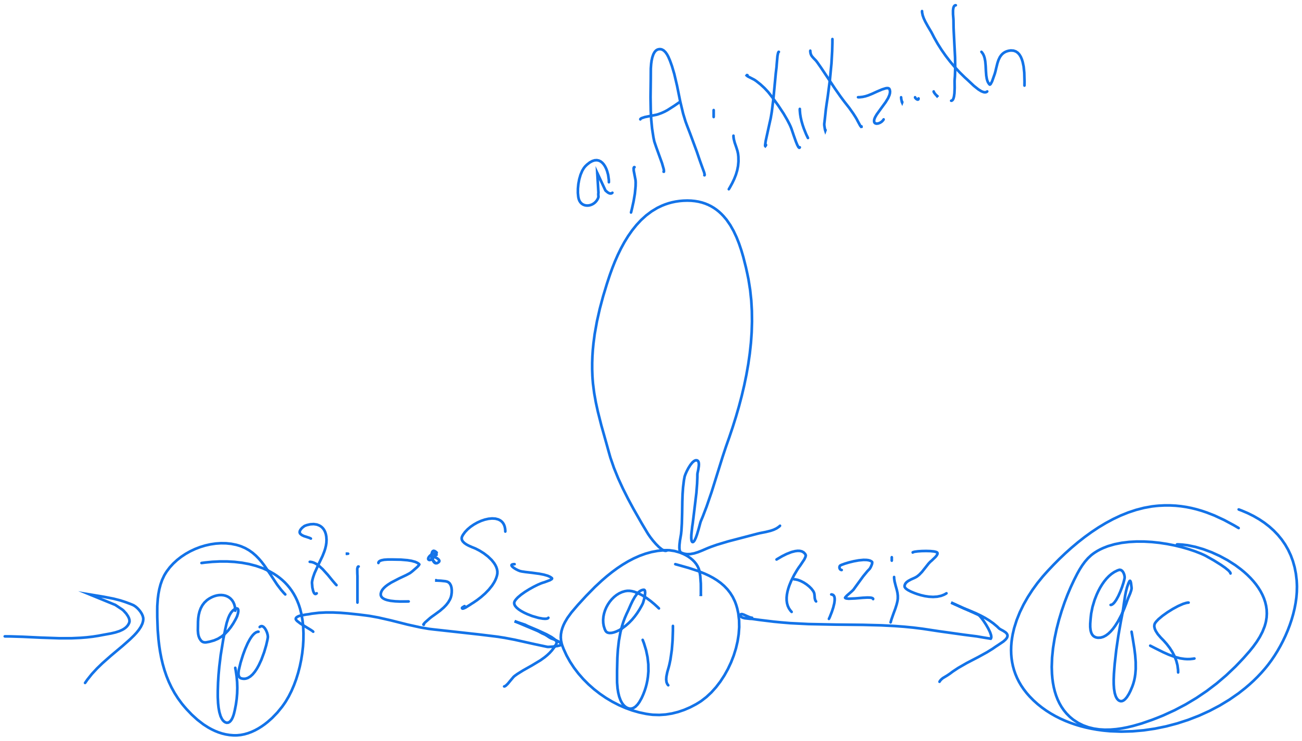
1. M starts by putting S on the stack

2. For each production

$A \to aX_1X_2\ldots X_n$

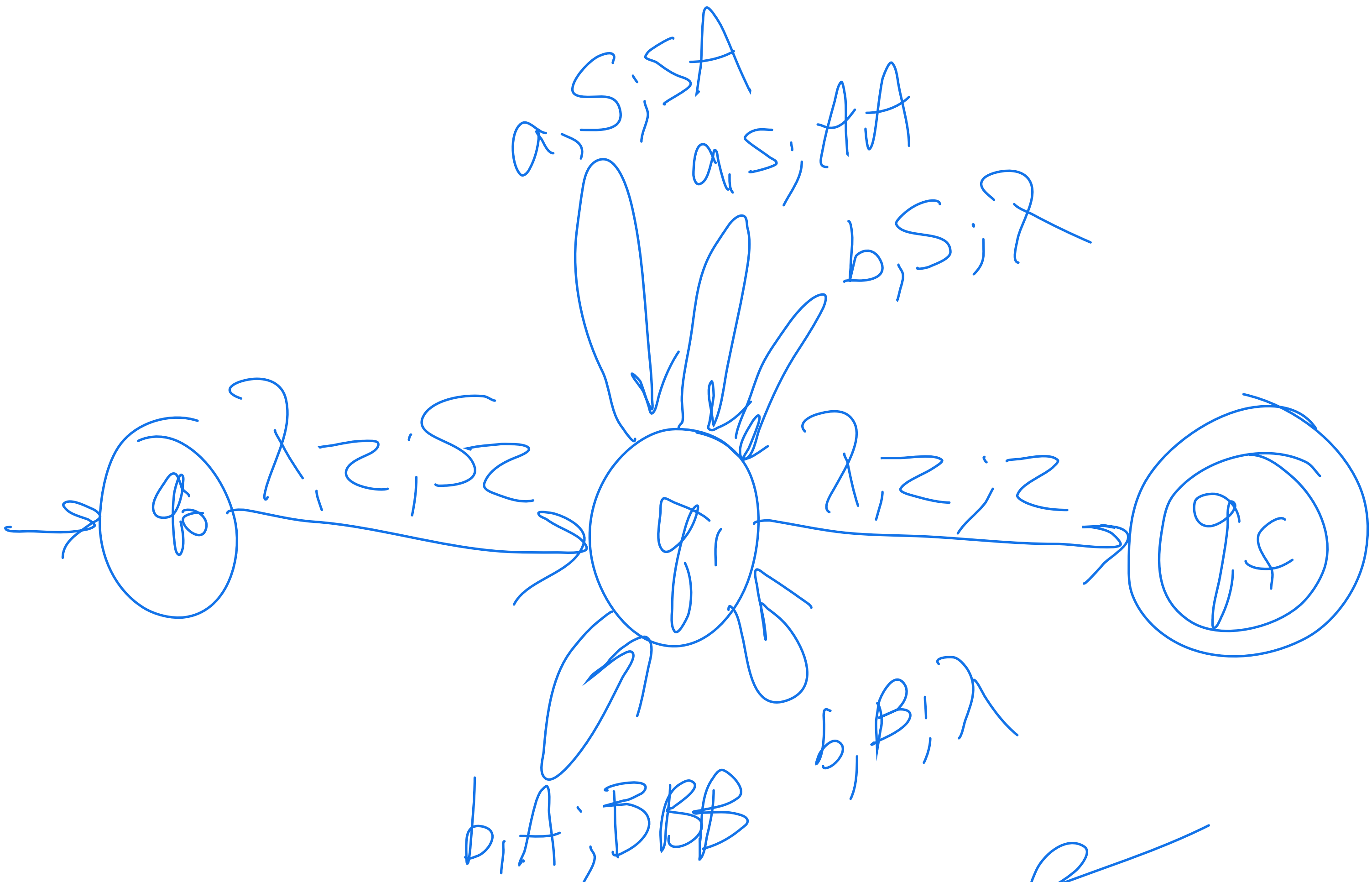
put $(q_1, X_1X_2\ldots X_n)$ in $\delta(q_1, a, A)$

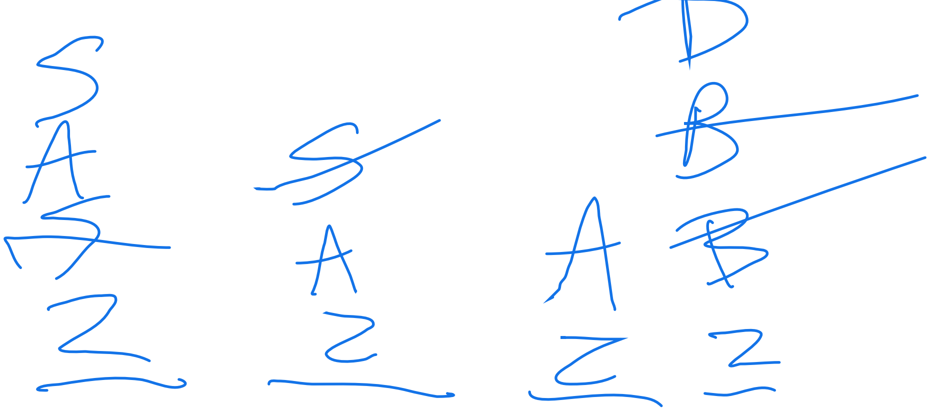
3. accept if $S \overset{*}{\Rightarrow} w$



Example: Let G'=(V,T,S,P), P=

$$\begin{aligned} \mathbf{S} &\rightarrow \mathbf{aSA} \mid \mathbf{aAA} \mid \mathbf{b} \\ \mathbf{A} &\rightarrow \mathbf{bBBB} \\ \mathbf{B} &\rightarrow \mathbf{b} \end{aligned}$$





~~abbbbb~~

Theorem Given a NPDA M, ∃ a NPDA M' s.t. all transitions have the form $\delta(q_i$,a,A)=$\{c_1, c_2, \ldots c_n\}$ where

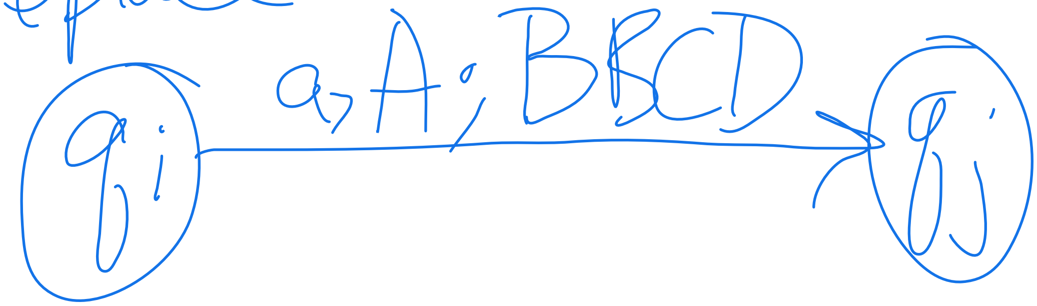
$$c_i=(q_j, \lambda)$$

or $c_i=(q_j$,**BC)**

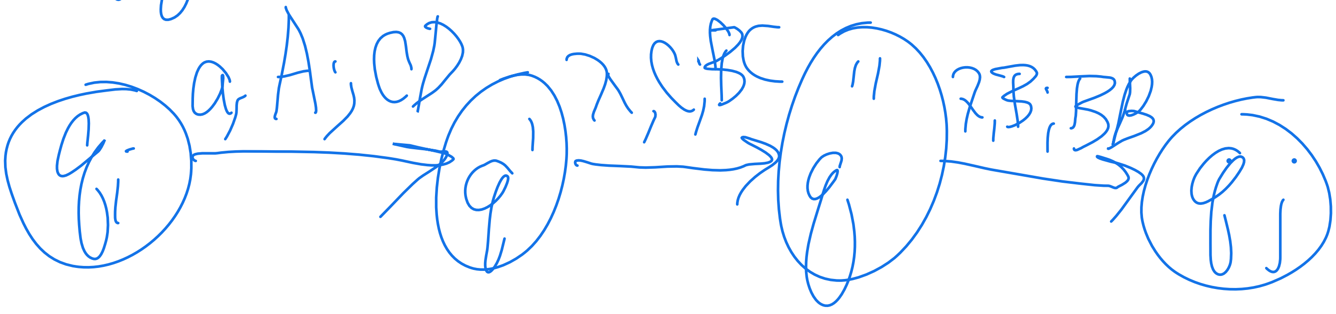
Each move either increases or decreases stack contents by a single symbol.

- **Proof (sketch)**

replace

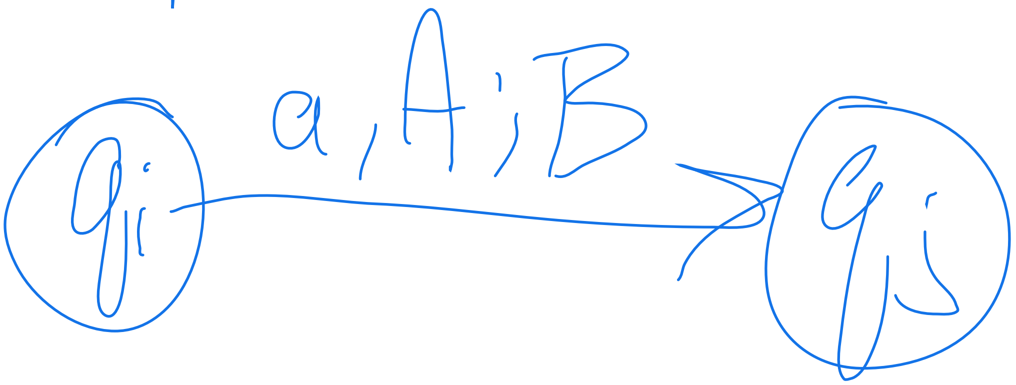


by

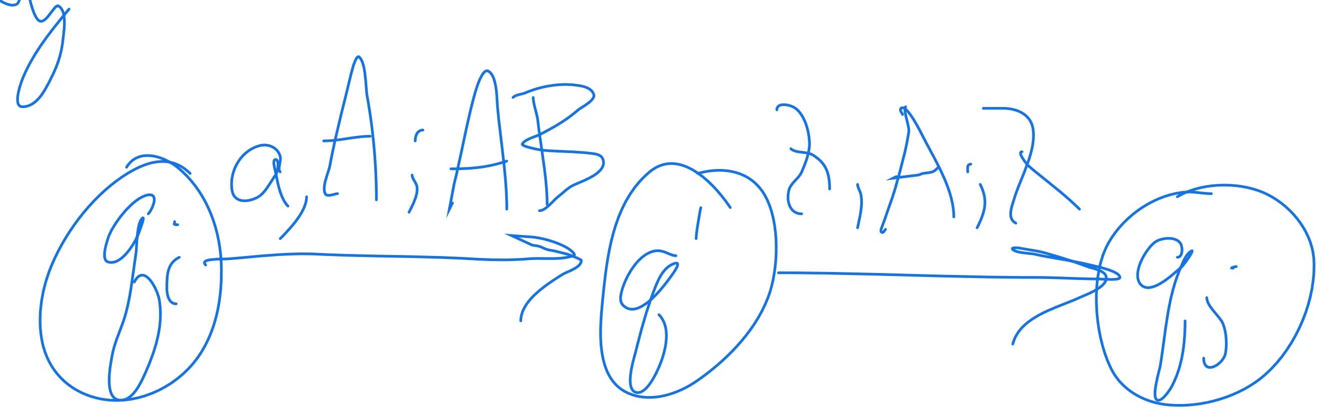


replace

same size



by



Theorem If L=L(M) for some NPDA M, then L is a CFL.

- **Proof: Given NPDA M.**

  First, construct an equivalent NPDA M that will be easier to work with. Construct M' such that

  1. **accepts if stack is empty**
  2. **each move increases or decreases stack content by a single symbol. (can only push 2 variables or no variables with each transition)**

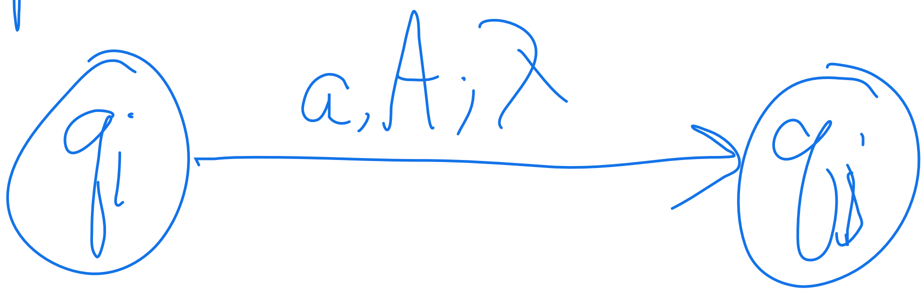
  M'=(Q,Σ**,**Γ**,**δ**,**q_0**,z,F)**

  Construct G=(V,Σ**,S,P) where**

  V=$\{(q_i c q_j) | q_i, q_j \in Q, c \in \Gamma\}$

  Goal:$S = (q_0 z q_f)$ **which will be the start symbol in the grammar.**

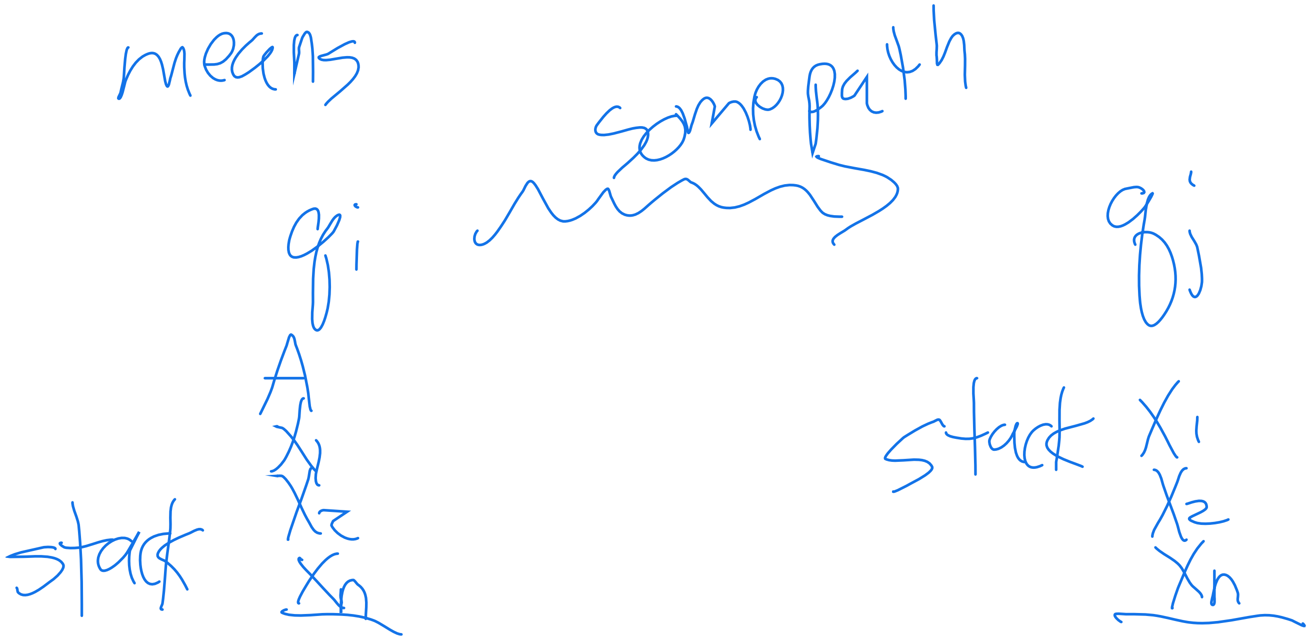
1) Replace



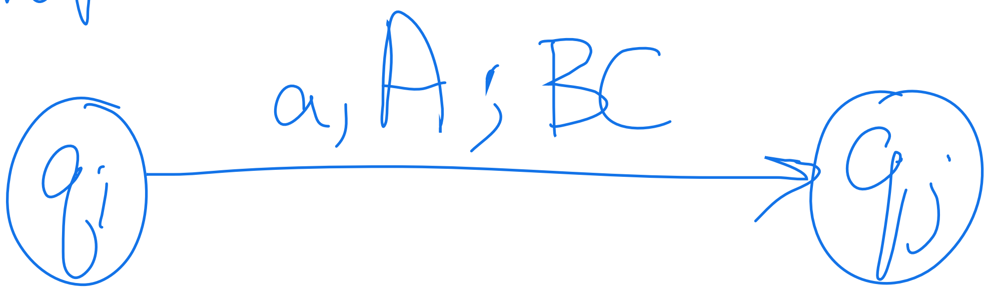
by

$(q_i A q_j) \to a$

means



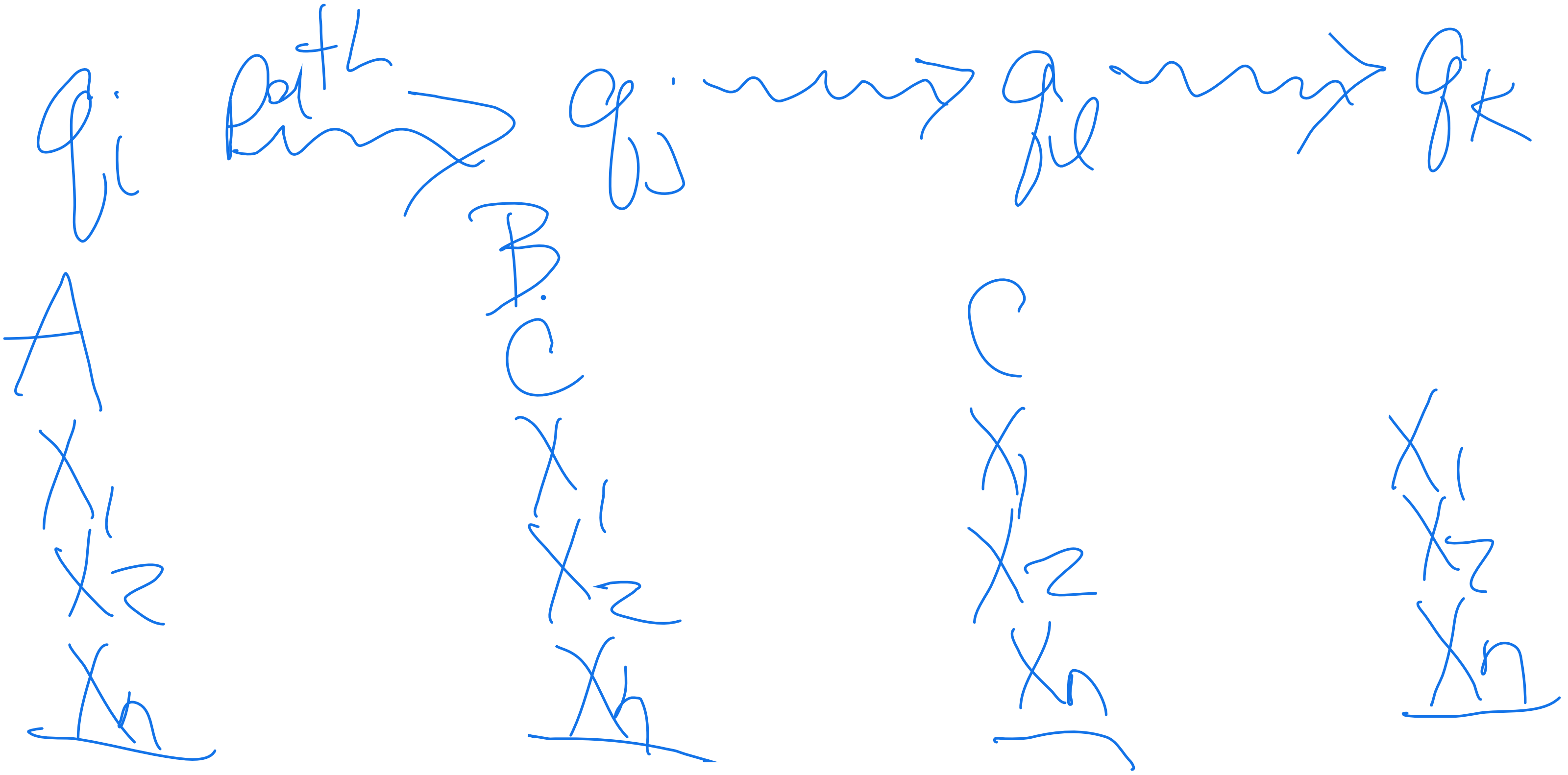
2) Replace



by

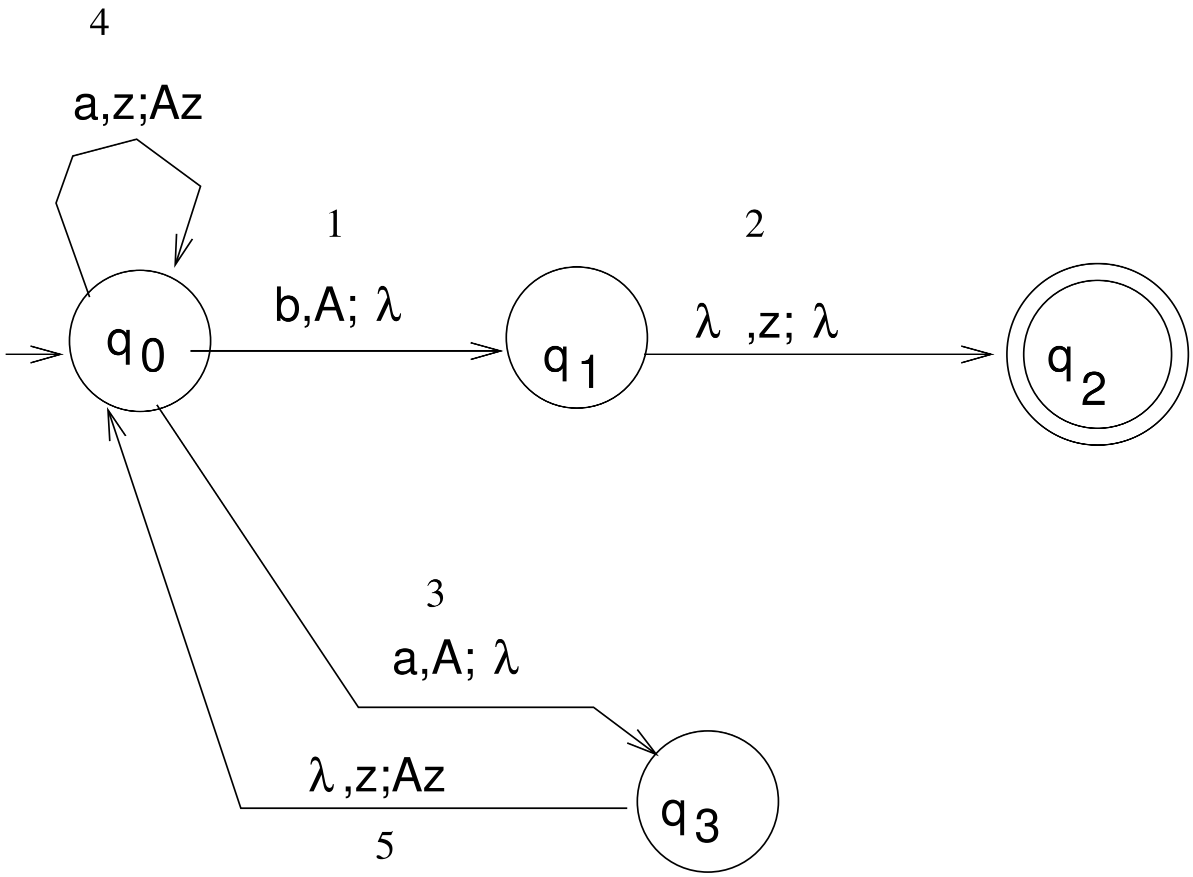
$$(q_i A q_k) \to a(q_j B q_l)(q_l C q_k)$$

for all $q_l, q_k \in Q$



Example:

L(M)=$\{aa^*b\}$, **M=(Q,**Σ**,**Γ**,**δ**,**q_0**,z,F),**
Q=$\{q_0, q_1, q_2, q_3\}$**,**
Σ**=**$\{a, b\}$**,**Γ**=**$\{A, z\}$**,F=**$\{\}$**.**



Construct the grammar G=(V,T,S,P),

V=$\{(q_0Aq_0),(q_0zq_0),(q_0Aq_1),(q_0zq_1),\ldots\}$

T=Σ

S=(q_0zq_2)

P=

From transition 1 $(q_0Aq_1) \rightarrow\ b$

From transition 2 $(q_1zq_2) \rightarrow\ \lambda$

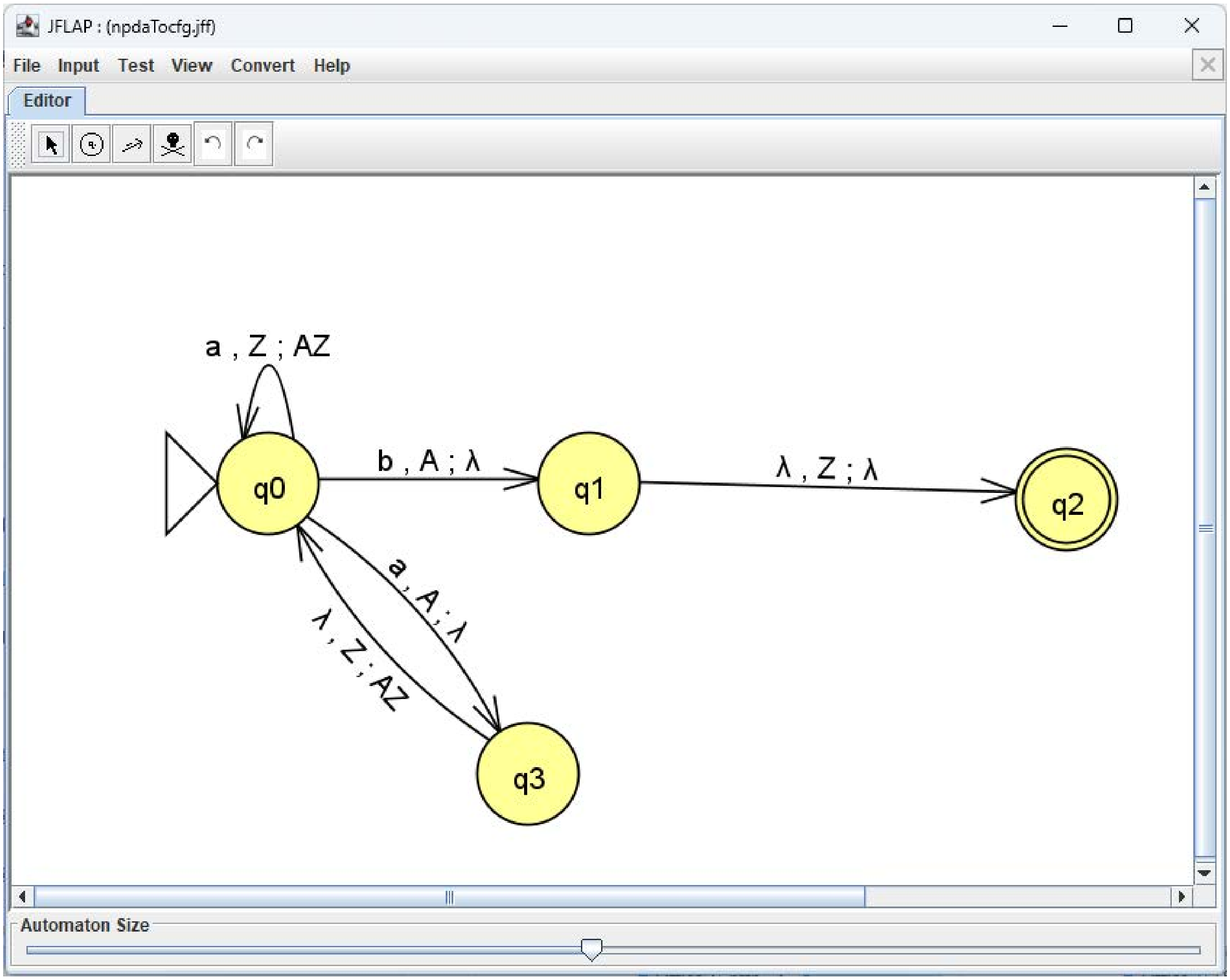
From transition 3 $(q_0Aq_3) \rightarrow\ a$

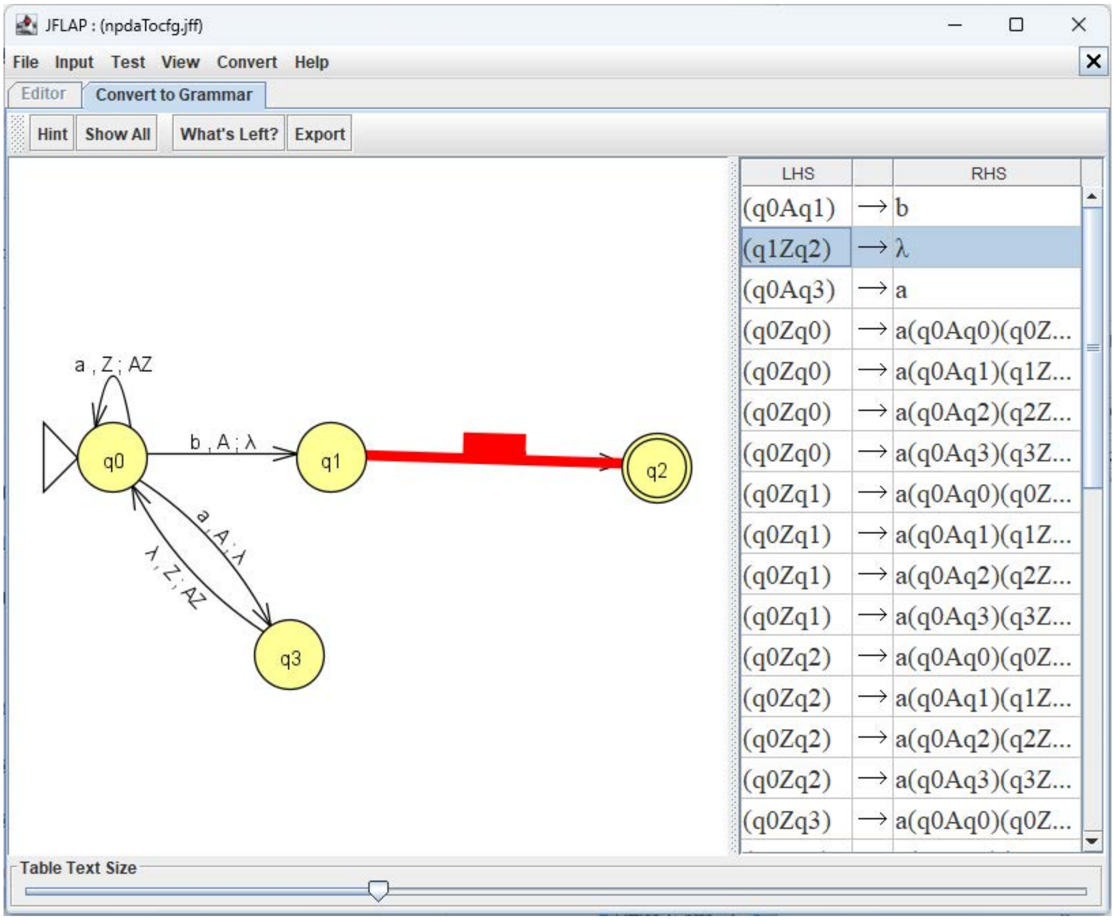
From transition 4
$$
\begin{aligned}
(q_0zq_0) \rightarrow\ & a(q_0Aq_0)(q_0zq_0)| \\
& a(q_0Aq_1)(q_1zq_0)| \\
& a(q_0Aq_2)(q_2zq_0)| \\
& a(q_0Aq_3)(q_3zq_0) \\
(q_0zq_1) \rightarrow\ & a(q_0Aq_0)(q_0zq_1)| \\
& a(q_0Aq_1)(q_1zq_1)| \\
& a(q_0Aq_2)(q_2zq_1)| \\
& a(q_0Aq_3)(q_3zq_1) \\
(q_0zq_2) \rightarrow\ & a(q_0Aq_0)(q_0zq_2)| \\
& a(q_0Aq_1)(q_1zq_2)| \\
& a(q_0Aq_2)(q_2zq_2)| \\
& a(q_0Aq_3)(q_3zq_2) \\
(q_0zq_3) \rightarrow\ & a(q_0Aq_0)(q_0zq_3)| \\
& a(q_0Aq_1)(q_1zq_3)| \\
& a(q_0Aq_2)(q_2zq_3)| \\
& a(q_0Aq_3)(q_3zq_3)
\end{aligned}
$$

From transition 5

$$\begin{aligned}
(q_3zq_0) \rightarrow\ & (q_0Aq_0)(q_0zq_0)| \\
& (q_0Aq_1)(q_1zq_0)| \\
& (q_0Aq_2)(q_2zq_0)| \\
& (q_0Aq_3)(q_3zq_0) \\
(q_3zq_1) \rightarrow\ & (q_0Aq_0)(q_0zq_1)| \\
& (q_0Aq_1)(q_1zq_1)| \\
& (q_0Aq_2)(q_2zq_1)| \\
& (q_0Aq_3)(q_3zq_1) \\
(q_3zq_2) \rightarrow\ & (q_0Aq_0)(q_0zq_2)| \\
& (q_0Aq_1)(q_1zq_2)| \\
& (q_0Aq_2)(q_2zq_2)| \\
& (q_0Aq_3)(q_3zq_2) \\
(q_3zq_3) \rightarrow\ & (q_0Aq_0)(q_0zq_3)| \\
& (q_0Aq_1)(q_1zq_3)| \\
& (q_0Aq_2)(q_2zq_3)| \\
& (q_0Aq_3)(q_3zq_3)
\end{aligned}$$

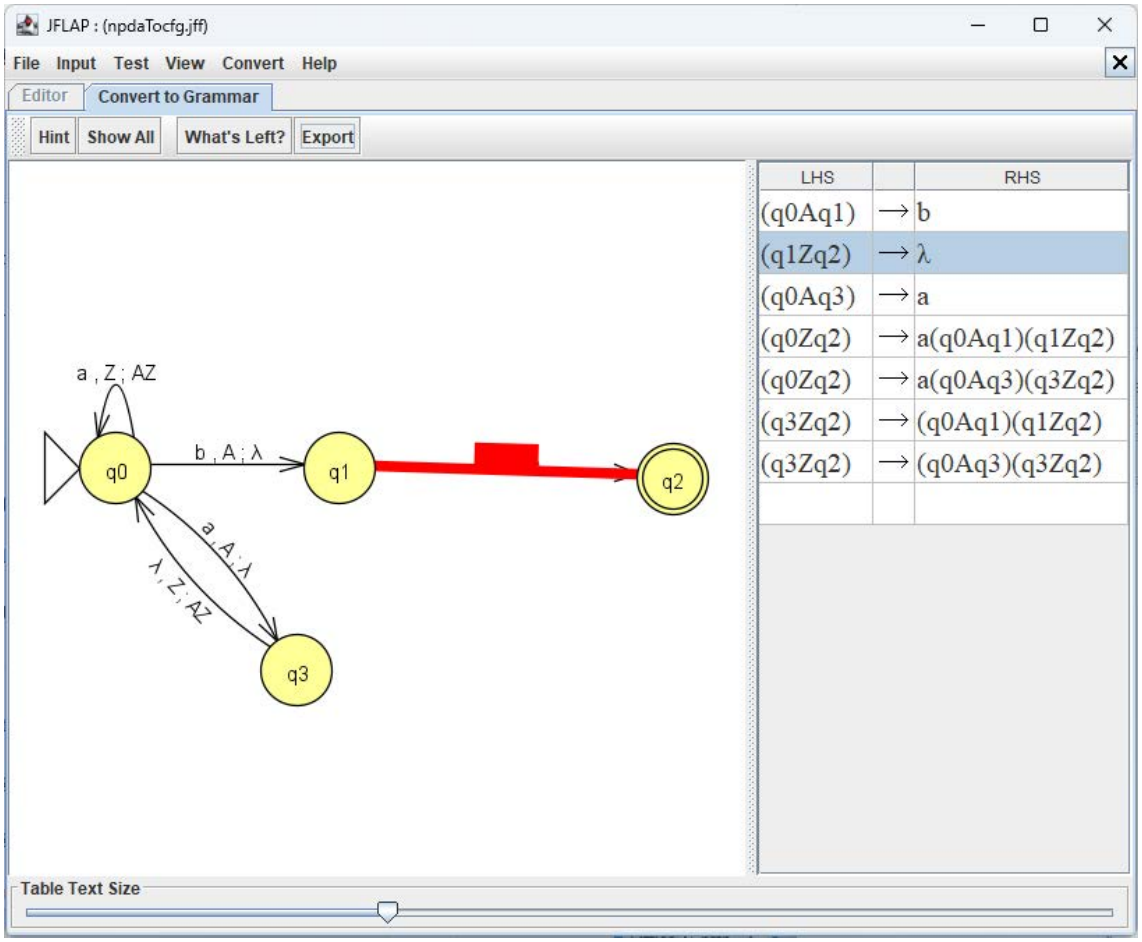
The NPDA in JFLAP





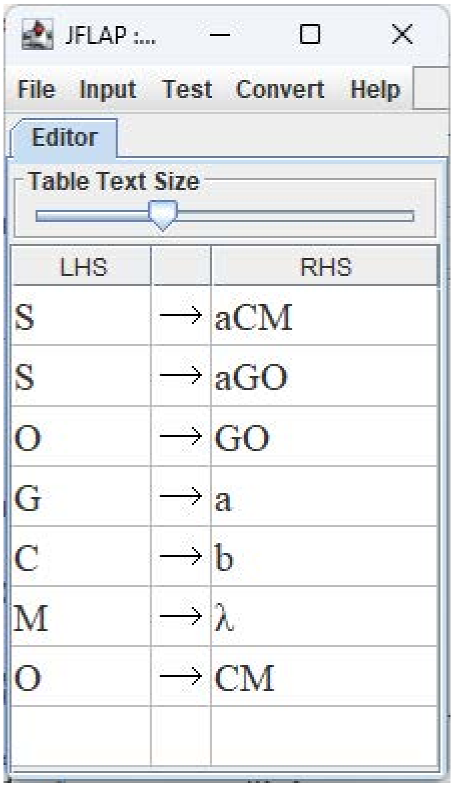
LHS		RHS
(q0Aq1)	→	b
(q1Zq2)	→	λ
(q0Aq3)	→	a
(q0Zq0)	→	a(q0Aq0)(q0Z...
(q0Zq0)	→	a(q0Aq1)(q1Z...
(q0Zq0)	→	a(q0Aq2)(q2Z...
(q0Zq0)	→	a(q0Aq3)(q3Z...
(q0Zq1)	→	a(q0Aq0)(q0Z...
(q0Zq1)	→	a(q0Aq1)(q1Z...
(q0Zq1)	→	a(q0Aq2)(q2Z...
(q0Zq1)	→	a(q0Aq3)(q3Z...
(q0Zq2)	→	a(q0Aq0)(q0Z...
(q0Zq2)	→	a(q0Aq1)(q1Z...
(q0Zq2)	→	a(q0Aq2)(q2Z...
(q0Zq2)	→	a(q0Aq3)(q3Z...
(q0Zq3)	→	a(q0Aq0)(q0Z...

The equivalent grammar in JFLAP, not all shown



LHS		RHS
(q0Aq1)	→	b
(q1Zq2)	→	λ
(q0Aq3)	→	a
(q0Zq2)	→	a(q0Aq1)(q1Zq2)
(q0Zq2)	→	a(q0Aq3)(q3Zq2)
(q3Zq2)	→	(q0Aq1)(q1Zq2)
(q3Zq2)	→	(q0Aq3)(q3Zq2)

The grammar trimmed and then exported, and variables converted to letters



LHS		RHS
S	→	aCM
S	→	aGO
O	→	GO
G	→	a
C	→	b
M	→	λ
O	→	CM

Recognizing aaab in M:

$$\begin{aligned}(q_0, aaab, z) &\vdash (q_0, aab, Az)\\ &\vdash (q_3, ab, z)\\ &\vdash (q_0, ab, Az)\\ &\vdash (q_3, b, z)\\ &\vdash (q_0, b, Az)\\ &\vdash (q_1, \lambda, z)\\ &\vdash (q_2, \lambda, \lambda)\end{aligned}$$

Derivation of string aaab in G:

$$\begin{aligned}(q_0 z q_2) &\Rightarrow a(q_0 A q_3)(q_3 z q_2)\\ &\Rightarrow aa(q_3 z q_2)\\ &\Rightarrow aa(q_0 A q_3)(q_3 z q_2)\\ &\Rightarrow aaa(q_3 z q_2)\\ &\Rightarrow aaa(q_0 A q_1)(q_1 z q_2)\\ &\Rightarrow aaab(q_1 z q_2)\\ &\Rightarrow aaab\end{aligned}$$

Stopped here

Chapter 7.3

Definition: A PDA M=(Q,Σ,Γ,δ,q_0,z,F) is *deterministic* **if for every** q $\in$**Q,** $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ **contains at most 1 element**
2. **if** $\delta(q, \lambda, b) \neq \emptyset$ **then** $\delta(q, c, b)=\emptyset$ **for all** $c \in \Sigma$

Definition: L is DCFL iff $\exists$ **DPDA M s.t. L=L(M).**

Examples:

1. **Previous pda for** $\{a^nb^n|n \geq 0\}$ **is deterministic.**
2. **Previous pda for** $\{a^nb^mc^{n+m}|n, m > 0\}$ **is deterministic.**
3. **Previous pda for** $\{ww^R|w \in \Sigma^+\}$,$\Sigma = \{a, b\}$ **is nondeterministic.**

Note: There are CFL's that are not deterministic.

L=$\{a^nb^n|n \geq 1\} \cup \{a^nb^{2n}|n \geq 1\}$ **is a CFL and not a DCFL.**

- **Proof:**
  $L = \{a^nb^n : n \geq 1\} \cup \{a^nb^{2n} : n \geq 1\}$
  It is easy to construct a NPDA for $\{a^nb^n : n \geq 1\}$ **and a NPDA for** $\{a^nb^{2n} : n \geq 1\}$**. These two can be joined together by a new start state**

and λ-transitions to create a NPDA for L. Thus, L is CFL.

Now show L is not a DCFL. Assume that there is a deterministic PDA M such that $L = L(M)$. We will construct a PDA that recognizes a language that is not a CFL and derive a contradiction.

Construct a PDA M' as follows:

1. **Create two copies of M: M_1 and M_2. The same state in M_1 and M_2 are called cousins.**
2. **Remove accept status from accept states in M_1, remove initial status from initial state in M_2. In our new PDA, we will start in M_1 and accept in M_2.**
3. **Outgoing arcs from old accept states in M_1, change to end up in the cousin of its destination in**

M_2. This joins M_1 and M_2 into one PDA. There must be an outgoing arc since you must recognize both a^nb^n and a^nb^{2n}. After reading n b's, must accept if no more b's and continue if there are more b's.

4. **Modify all transitions that read a b and have their destinations in M_2 to read a c.**

This is the construction of our new PDA.

When we read a^nb^n and end up in an old accept state in M_1, then we will transfer to M_2 and read the rest of a^nb^{2n}. Only the b's in M_2 have been replaced by c's, so the new machine accepts $a^nb^nc^n$.

The language accepted by our new PDA is $a^nb^nc^n$. But this is not a CFL. Contradiction! Thus there is

b
c
b
c

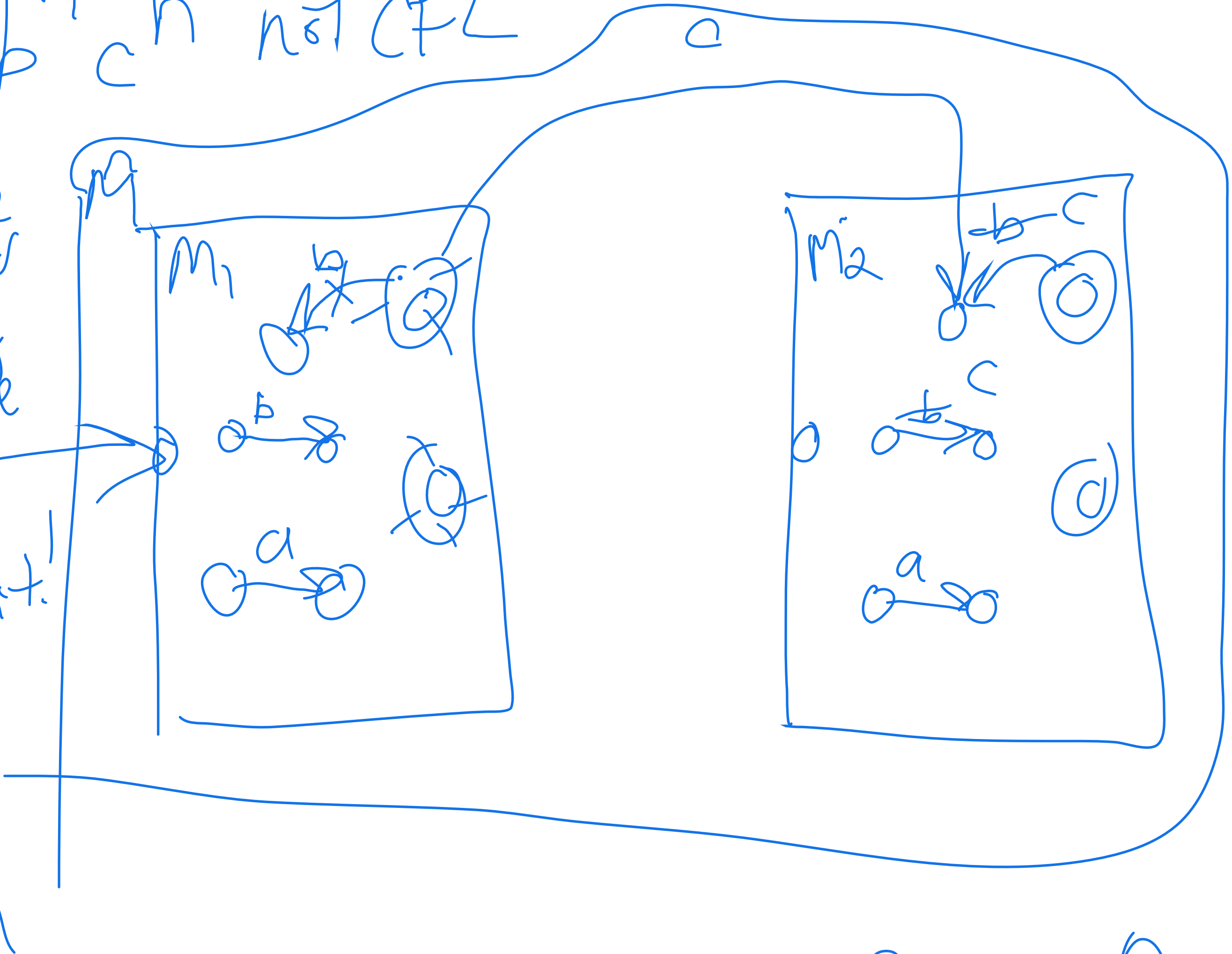
no deterministic PDA M such that $L(M) = L$. Q.E.D.

aabb. aaabbb

aabbbb aaabbbbbb

$a^n b^n c^n$ not CFL

Will prove this later but try to create an NPDA for it! you can't!



Λ

M recognizes $a^n b^n c^n$ which is not a CFL. Can't design an npda for that. ~~haven~~ $\Rightarrow$ Contradiction $\Rightarrow$ M is not an DPDA.