9/10/2024

Section: Regular Languages

Regular Expressions

Method to represent strings in a language

$+$ **union (or)**
$\circ$ **concatenation (AND) (can omit)**
$*$ **star-closure (repeat 0 or more times)**

Example: $\Sigma = \{a, b\}$

$(a+b)^* \circ a \circ (a+b)^* = (a+b)^* a (a+b)^*$

strings over Σ^* that contain at least one a

Example: $\Sigma = \{a\}$

$(aa)^*$ even no. of a's

Definition Given Σ,

1. $\emptyset$, λ, $a \in \Sigma$ are R.E.
2. If r and s are R.E. then
   - r+s is R.E.
   - rs is R.E.
   - (r) is a R.E.
   - r^* is R.E.
3. r is a R.E. iff it can be derived from (1) with a finite number of applications of (2).

Definition: L(r) = language denoted by R.E. r.

1. $\emptyset$, $\{\lambda\}$, $\{a\}$ **are L denoted by a R.E.**
2. **if r and s are R.E. then**
   (a) $\mathbf{L(r+s) = L(r) \cup L(s)}$
   (b) $\mathbf{L(rs) = L(r) \circ L(s)}$
   (c) $\mathbf{L((r)) = L(r)}$
   (d) $\mathbf{L((r)^*) = (L(r)^*)}$

Precedence Rules

$*$ **highest**

$\circ$

$+$

Example:

$ab^* + c =$ $\left(a\left(b^*\right)\right) + c$

same thing

Examples:

1. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ **has an odd number of** a**'s followed by an even number of** b**'s**$\}$.

$(aa)^* a (bb)^*$

2. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ **has no more than 3** a**'s and must end in** $ab\}$.

$b^*ab^*ab^*ab + b^*ab^*ab + b^*ab$

$b^*(a+\lambda)b^*(a+\lambda)b^*ab$

$b^*(ab^* + ab^*ab^* + \lambda)ab$

3. **Regular expression for all integers (including negative)**

$0 + (- + \lambda)((1+2+\dots+9)(0+1+2+\dots 9)^*)$

Section 3.2 Equivalence of DFA and R.E.

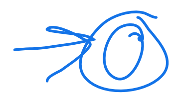
Theorem Let r be a R.E. Then $\exists$ NFA M s.t. L(M) = L(r).

- **Proof:**

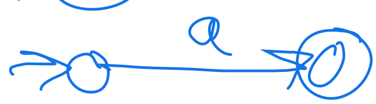
  $\emptyset$

  $\{\lambda\}$

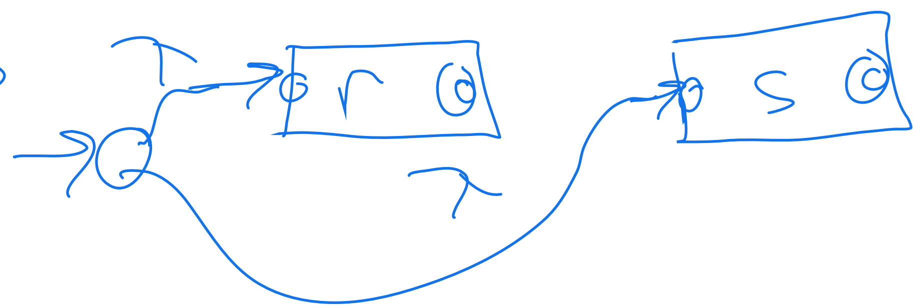
  

  $\{a\}$

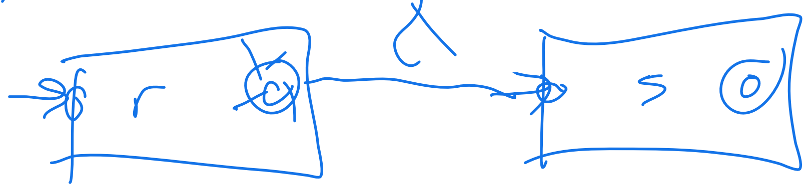
  Suppose r and s are R.E.

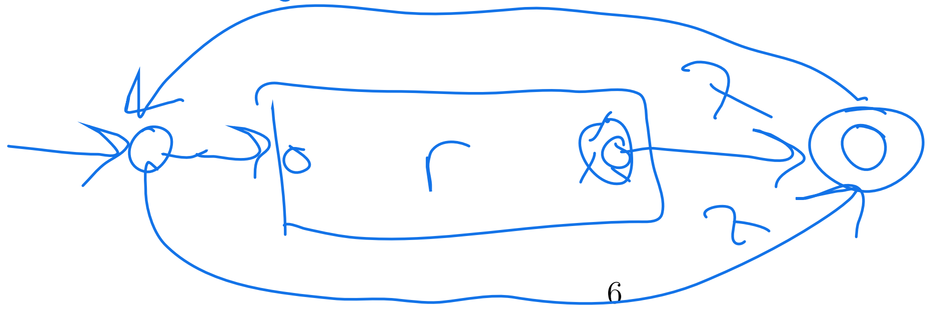
  1. r+s

  2. r$\circ$s

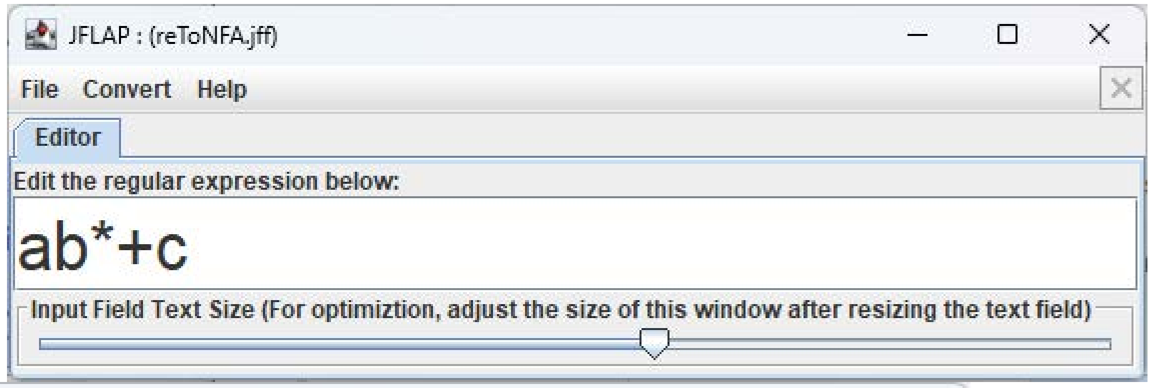
  3. r*



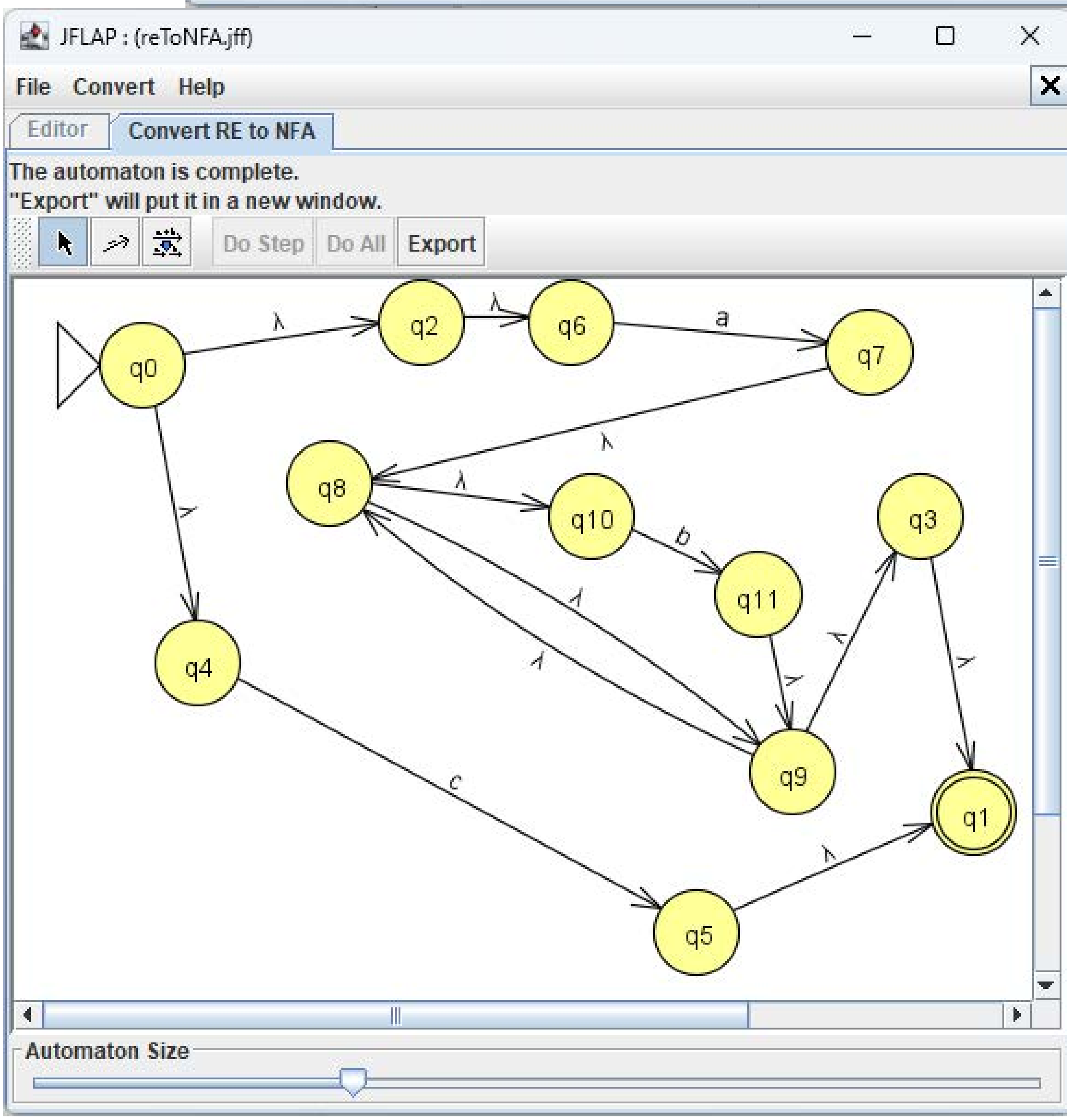


Example

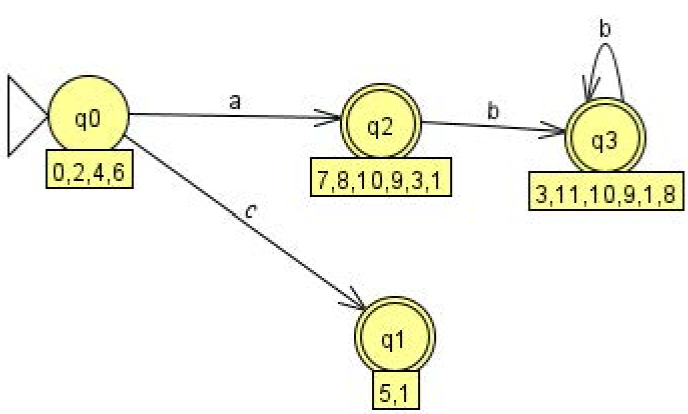
$ab^* + c$



Convert to NFA:



Then convert to DFA:



Theorem Let L be regular. Then $\exists$ R.E. r s.t. L=L(r).

Proof Idea: remove states sucessively until two states left

- **Proof:**

  L is regular

  $\Rightarrow \exists$ NFA M s.t. L = L(M)

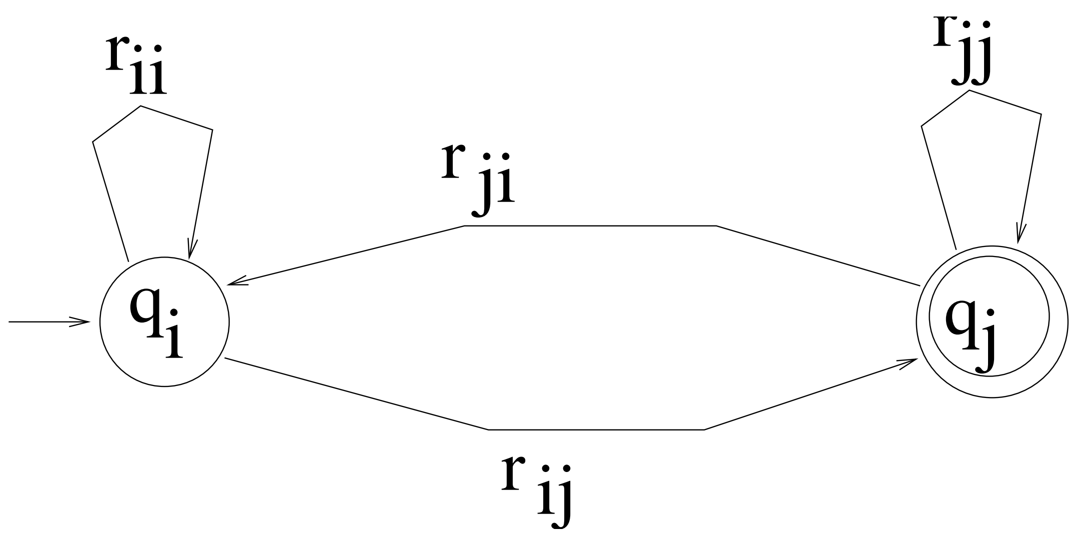
  1. Assume M has one final state and $q_0 \notin F$

  2. Convert to a generalized transition graph (GTG), all possible edges are present.

     If no edge, label with $\emptyset$

     Let r_{ij} stand for label of the edge from q_i to q_j

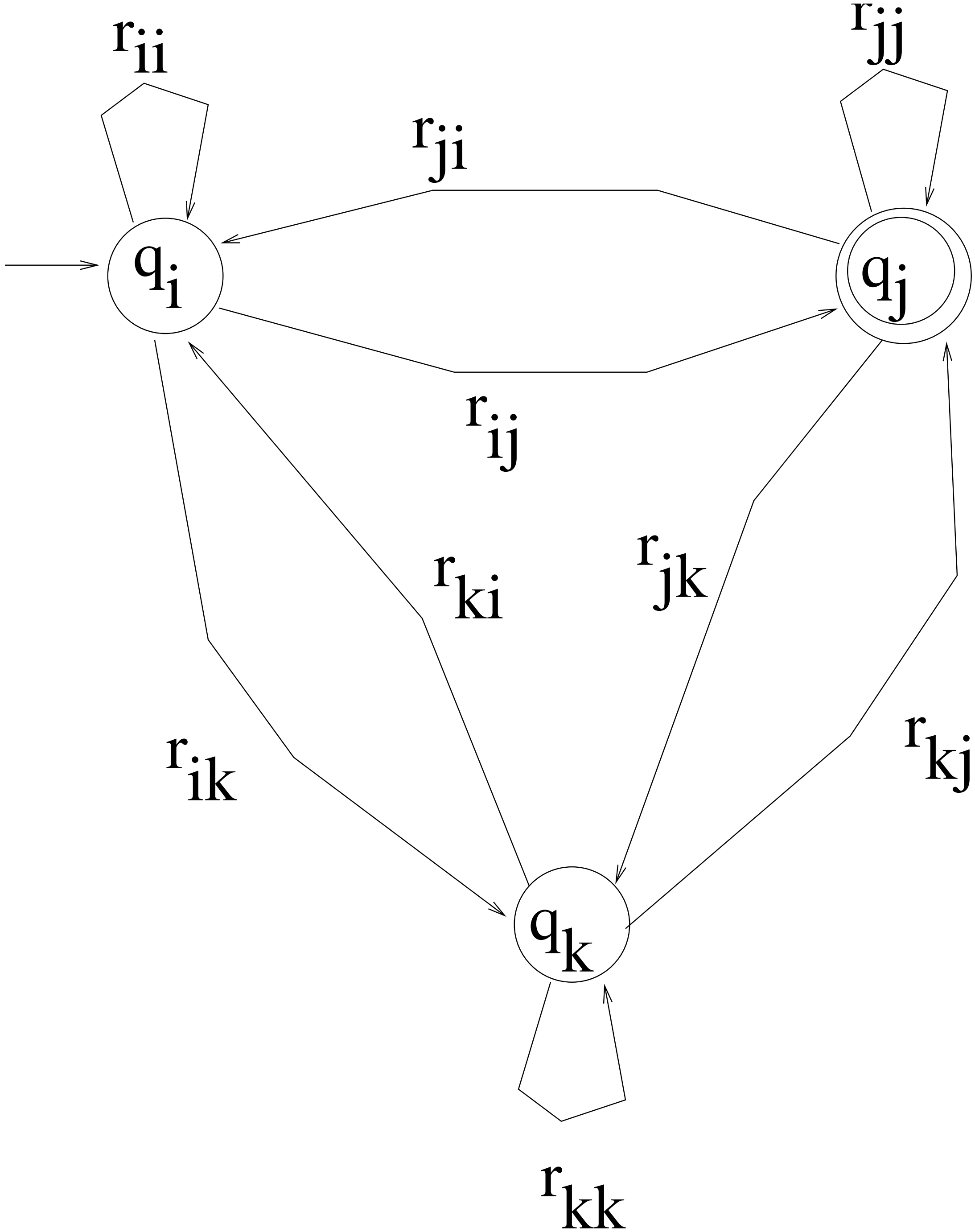
3. If the GTG has only two states, then it has the following form:



In this case the regular expression is:

$r = (r_{ii}^{*} r_{ij} r_{jj}^{*} r_{ji})^{*} r_{ii}^{*} r_{ij} r_{jj}^{*}$

4. If the GTG has three states then it must have the following form:



REPLACE	**WITH**
r_{ii}	$r_{ii} + r_{ik}r_{kk}^{*}r_{ki}$
r_{jj}	$r_{jj} + r_{jk}r_{kk}^{*}r_{kj}$
r_{ij}	$r_{ij} + r_{ik}r_{kk}^{*}r_{kj}$
r_{ji}	$r_{ji} + r_{jk}r_{kk}^{*}r_{ki}$

remove state q_k

5. If the GTG has four or more states, pick a state q_k to be removed (not initial or final state).

For all $o \neq k, p \neq k$ use the rule r_{op} replaced with $r_{op} + r_{ok}r_{kk}^*r_{kp}$ with different values of o and p.

When done, remove q_k and all its edges. Continue eliminating states until only two states are left. Finish with step 3.

6. In each step, simplify the regular expressions r and s with:

$r + r = r$

$s + r^* s = r^* s$

$r + \emptyset = r$

$r \emptyset = \emptyset$

$\emptyset^* = \{\lambda\}$

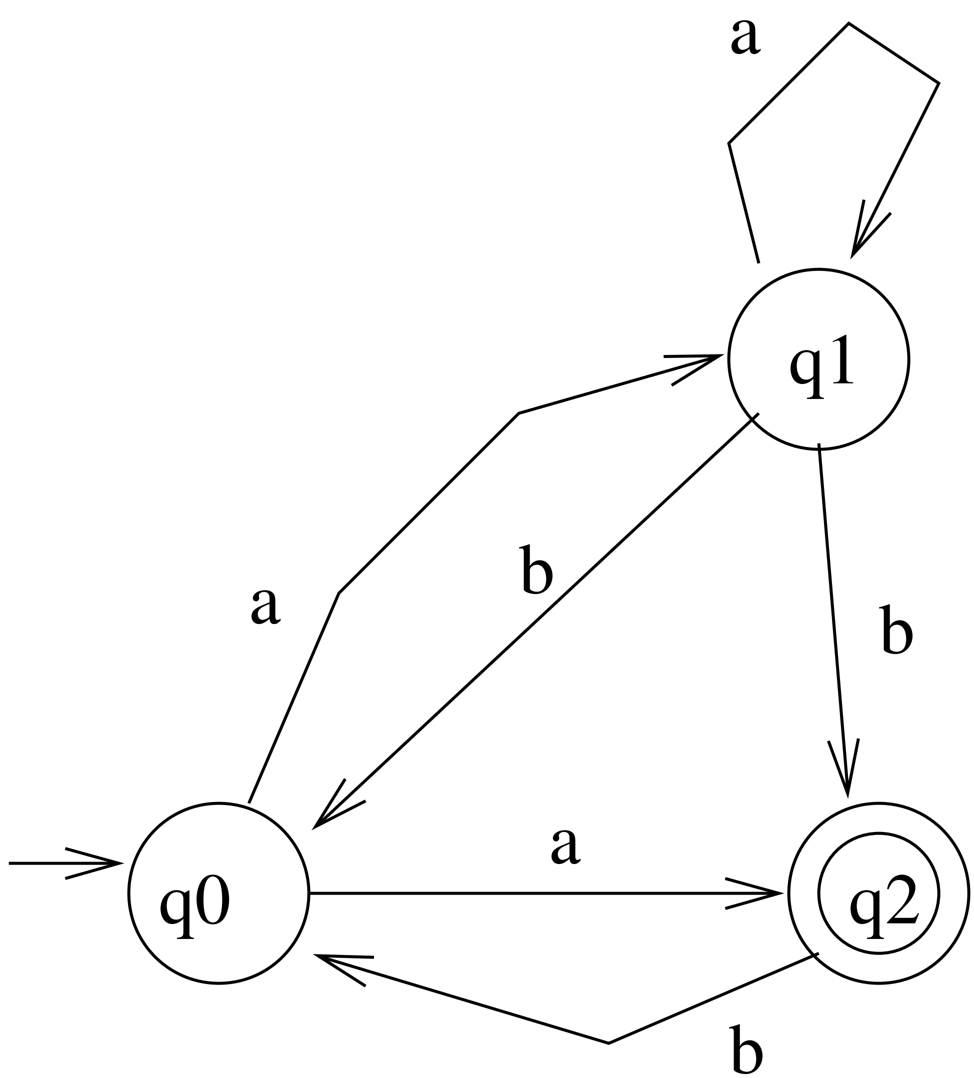
$r \lambda = r$

$(\lambda + r)^* = r^*$

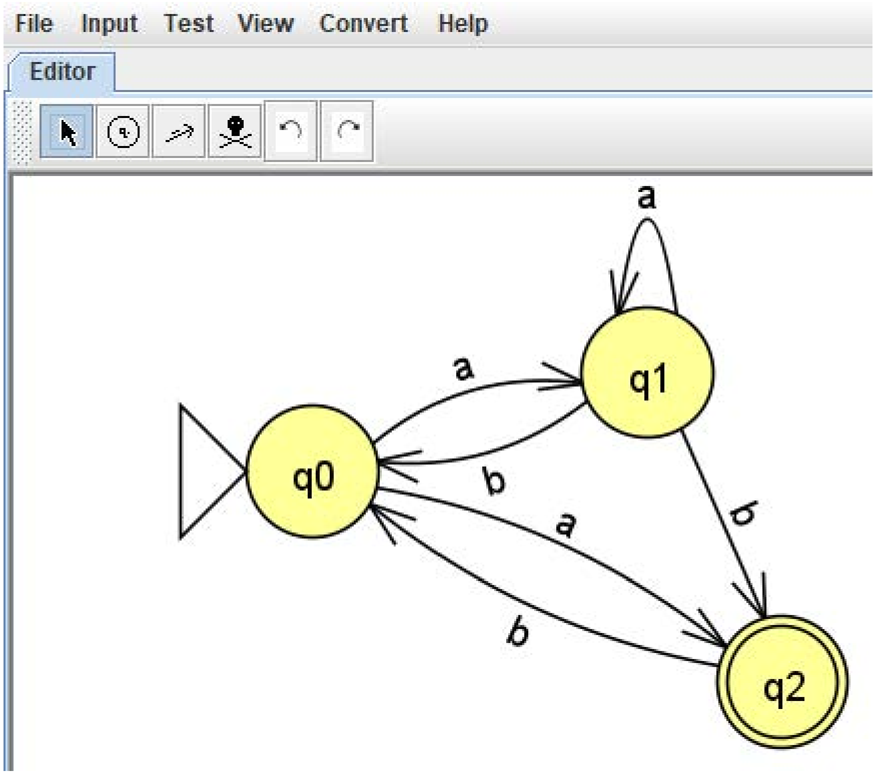
$(\lambda + r) r^* = r^*$

and similar rules.

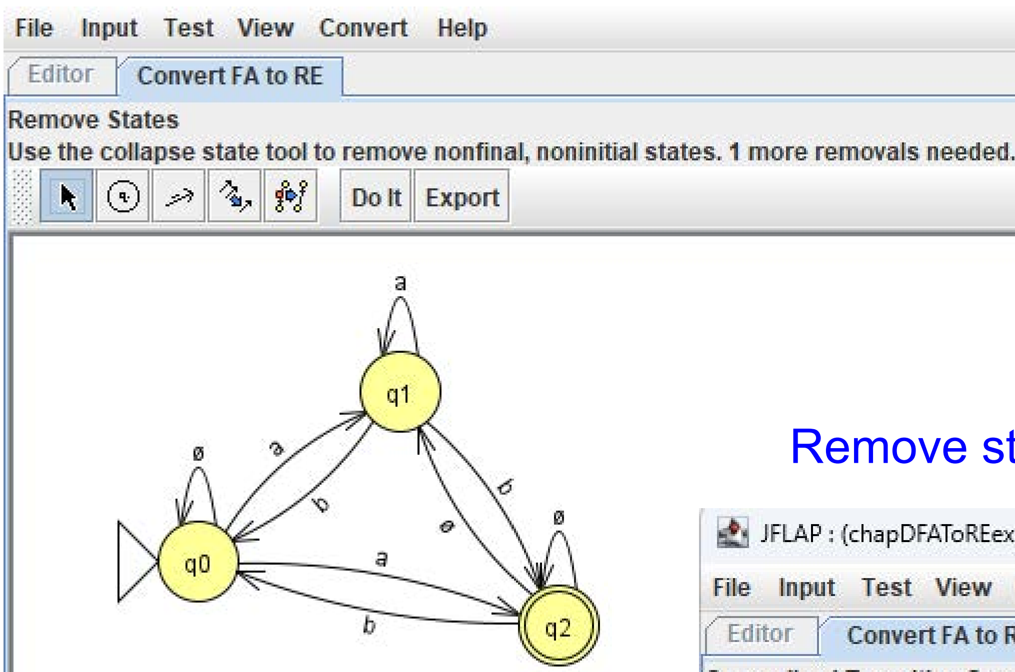
Example:



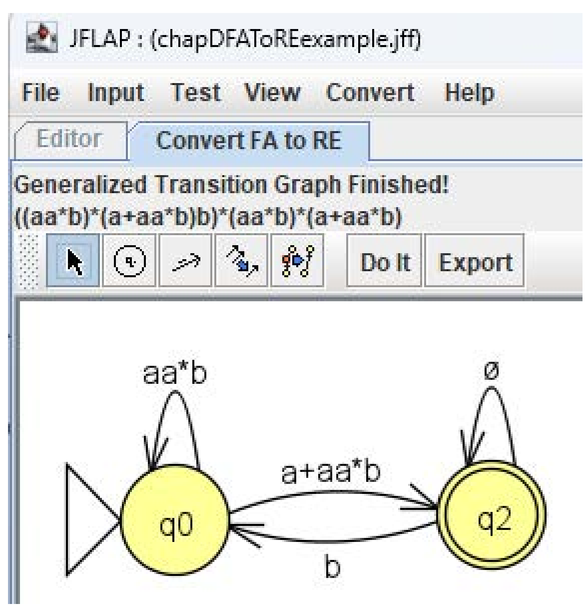
start with:



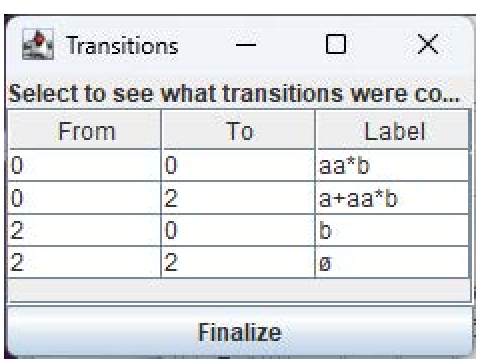
Add all arcs:



Remove state q1



New transitions when q1 removed:



From	To	Label
0	0	aa*b
0	2	a+aa*b
2	0	b
2	2	ø

Grammar G=(V,T,S,P)

V variables (nonterminals)
T terminals
S start symbol
P productions

Right-linear grammar:

all productions of form

$$\mathbf{A \rightarrow xB}$$

$$\mathbf{A \rightarrow x}$$

where $\mathbf{A,B \in V, x \in T^*}$

Left-linear grammar:

all productions of form

$$\mathbf{A} \rightarrow \mathbf{Bx}$$

$$\mathbf{A} \rightarrow \mathbf{x}$$

where $\mathbf{A,B} \in \mathbf{V}$, $\mathbf{x} \in \mathbf{T}^*$

Definition:

A regular grammar is a right-linear or left-linear grammar.

Example 1:

$$\mathbf{G=(\{S\},\{a,b\},S,P),\ P=}$$

$$\mathbf{S} \rightarrow \mathbf{ab\underline{S}}$$

$$\mathbf{S} \rightarrow \lambda$$

$$\mathbf{S} \rightarrow \mathbf{\underline{S}ab}$$

No, not regular

Example 2: regular grammar

$$\mathbf{G=(\{S,B\},\{a,b\},S,P),\ P=}$$
$$\mathbf{S \rightarrow aB \mid bS} \mid \lambda$$
$$\mathbf{B \rightarrow aS \mid bB}$$

L = { strings with an even no. of a's }

Theorem: L is a regular language iff $\exists$ regular grammar G s.t. L=L(G).

Outline of proof:

($\Longleftarrow$) Given a regular grammar G
Construct NFA M
Show L(G)=L(M)
($\Longrightarrow$) Given a regular language
$\exists$ DFA M s.t. L=L(M)
Construct reg. grammar G
Show L(G) = L(M)

Proof of Theorem:

($\Longleftarrow$) **Given a regular grammar G**
G=(V,T,S,P)

V=$\{V_0, V_1, \ldots, V_y\}$
T=$\{v_o, v_1, \ldots, v_z\}$
S=V_0

Assume G is right-linear
(see book for left-linear case).
Construct NFA M s.t. L(G)=L(M)
If w$\in$**L(G), w=**$v_1 v_2 \ldots v_k$

$$V_0 \Rightarrow v_1 V_i$$
$$\Rightarrow v_1 v_2 V_j$$
$$\overset{*}{\Rightarrow} v_1 v_2 \ldots v_{k-1} V_n$$
$$\Rightarrow v_1 v_2 \ldots v_{k-1} v_k$$

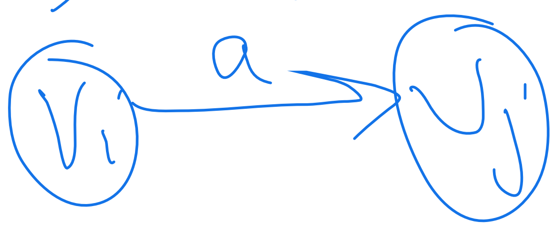
idea

M=(V$\cup\{V_f\}$,T,δ,V_0,$\{V_f\}$)

V_0 is the start (initial) state

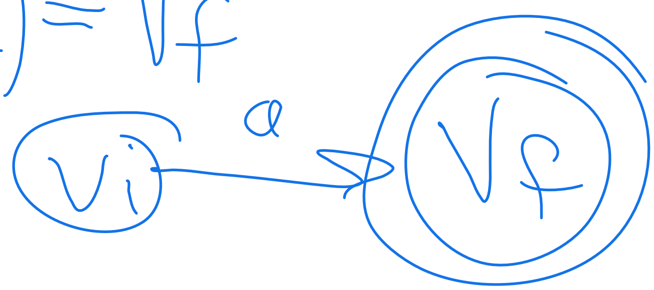
For each production, $V_i \to aV_j$,

$\delta(V_i, a) = V_j$



For each production, $V_i \to a$,

$\delta(V_i, a) = V_f$



Show L(G)=L(M)

Thus, given R.G. G,

L(G) is regular

($\Longrightarrow$) **Given a regular language L**

$\exists$ **DFA M s.t. L=L(M)**

M=(Q,Σ**,**δ**,**q_0**, F)**

Q=$\{q_0, q_1, \ldots, q_n\}$

$\Sigma = \{a_1, a_2, \ldots, a_m\}$

Construct R.G. G s.t. L(G) = L(M

G=(Q,Σ**,**q_0**,P)**

if $\delta(q_i, a_j)$**=**q_k **then**

$q_i \to a_j q_k$

if $q_k \in$**F then**

$q_k \to \lambda$

Show w $\in$**L(M)** $\Longleftrightarrow$ **w** $\in$ **L(G)**

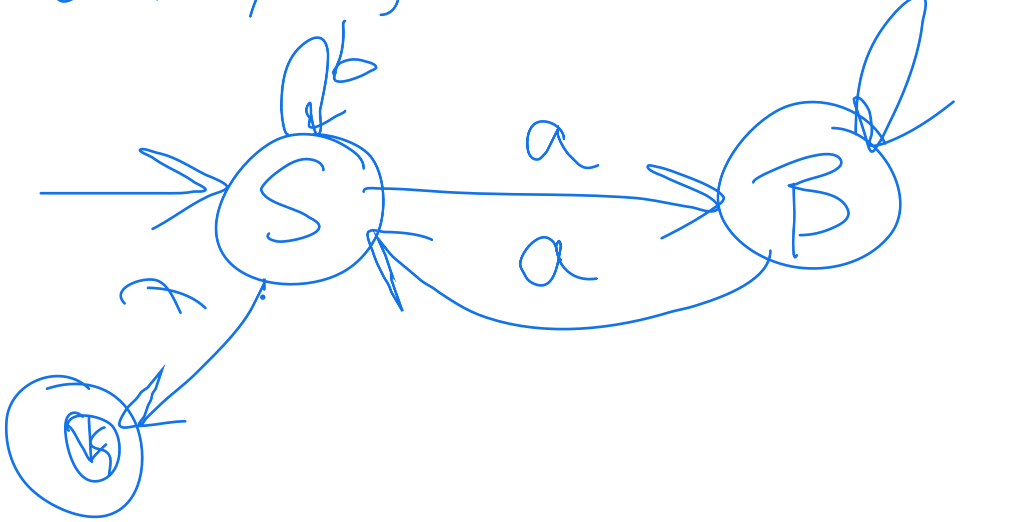
Thus, L(G)=L(M).

QED.

Example

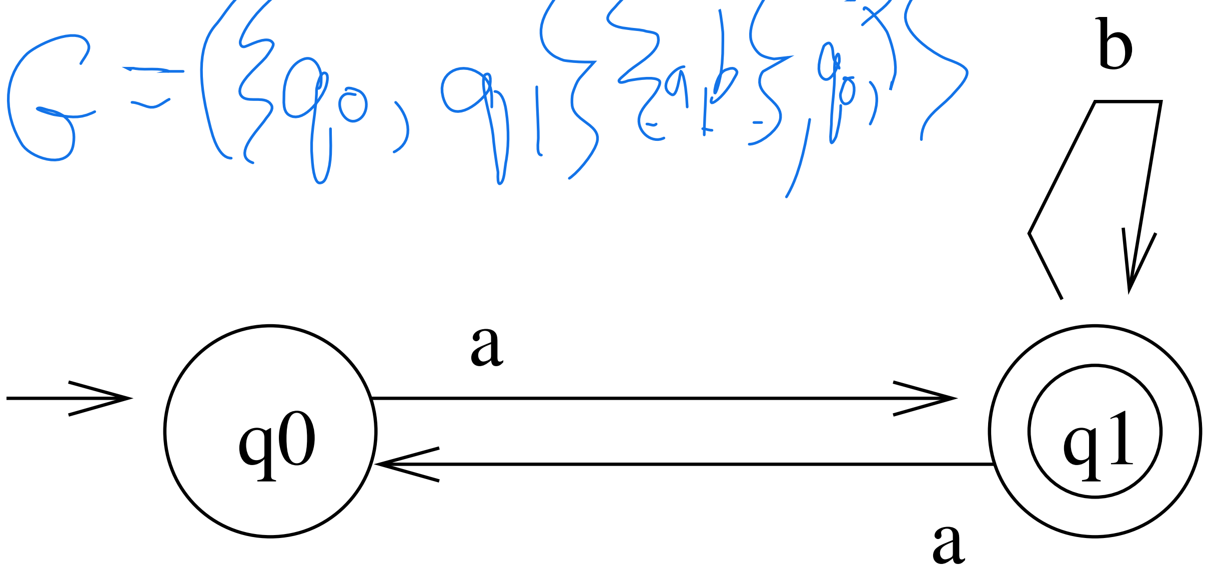
$$\mathbf{G=(\{S,B\},\{a,b\},S,P),\ P=}$$
$$\mathbf{S \rightarrow aB \mid bS \mid} \lambda$$
$$\mathbf{B \rightarrow aS \mid bB}$$

$Q = \{S, B, V_f\}$



Example:

$G = (\{q_0, q_1\}\{a, b\}, q_0, P)$



$P =$

$$q_0 \rightarrow a\, q_1$$
$$q_1 \rightarrow b\, q_1 \mid a\, q_0 \mid \varepsilon$$