Section: Properties of Regular Languages

Example

$L = \{a^n b a^n \mid n > 0\}$

Not regular!

$= \{aba, aabaa, \ldots\}$

$abaa \in L$ No

Can you build a DFA?
or regexpr?

Closure Properties

A set is closed over an operation if

$$\mathbf{L_1, L_2 \in class}$$
$$\mathbf{L_1\ op\ L_2 = L_3}$$
$$\mathbf{\Rightarrow L_3 \in class}$$

L={x | x is a positive even integer}

L is closed under

addition? yes
multiplication? yes
subtraction? no $6-10=-4 \notin L$
division? no

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

$$L_1 \cup L_2$$
$$L_1 \cap L_2$$
$$L_1 L_2$$
$$\bar{L}_1$$
$$L_1^*$$

are regular languages.

Proof(sketch)

L_1 and L_2 are regular languages
$\Rightarrow$ $\exists$ reg. expr. r_1 and r_2 s.t.
$L_1 = L(r_1)$ and L_2=$L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
r_1^* is r.e. denoting L_1^*
$\Rightarrow$ closed under star-closure

complementation:

L_1 is reg. lang.

$\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$

Construct M' s.t.

final states in M
are nonfinal state in M'
nonfinal states in M
are final states in M'

show $w \in L(M') \Leftrightarrow w \in \overline{L}$

$\Rightarrow$ closed under complementation

intersection:

$\mathbf{L}_1$ **and** $\mathbf{L}_2$ **are reg. lang.**

$\Rightarrow \exists$ **DFA** $\mathbf{M}_1$ **and** $\mathbf{M}_2$ **s.t.**

$\mathbf{L}_1 = \mathbf{L}(\mathbf{M}_1)$ **and** $\mathbf{L}_2 = \mathbf{L}(\mathbf{M}_2)$

$\mathbf{M}_1 = (\mathbf{Q}, \Sigma, \delta_1, q_0, \mathbf{F}_1)$

$\mathbf{M}_2 = (\mathbf{P}, \Sigma, \delta_2, p_0, \mathbf{F}_2)$

Construct $\mathbf{M}' = (\mathbf{Q}', \Sigma, \delta', (q_0, p_0), \mathbf{F}')$

$\mathbf{Q}' = Q \times P$

δ': $((q_i, p_j), a) = (q_k, p_l)$ if

$\delta_1(q_i, a) = q_k \in M_1$ and

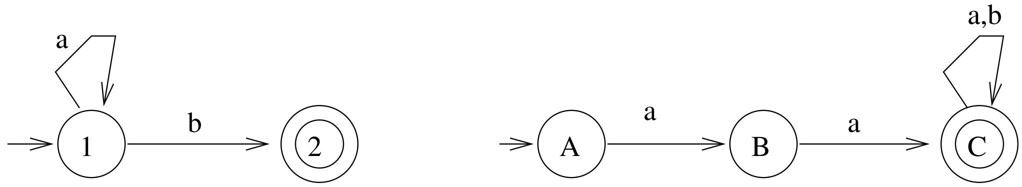
$\delta_2(p_j, a) = p_l \in M_2$

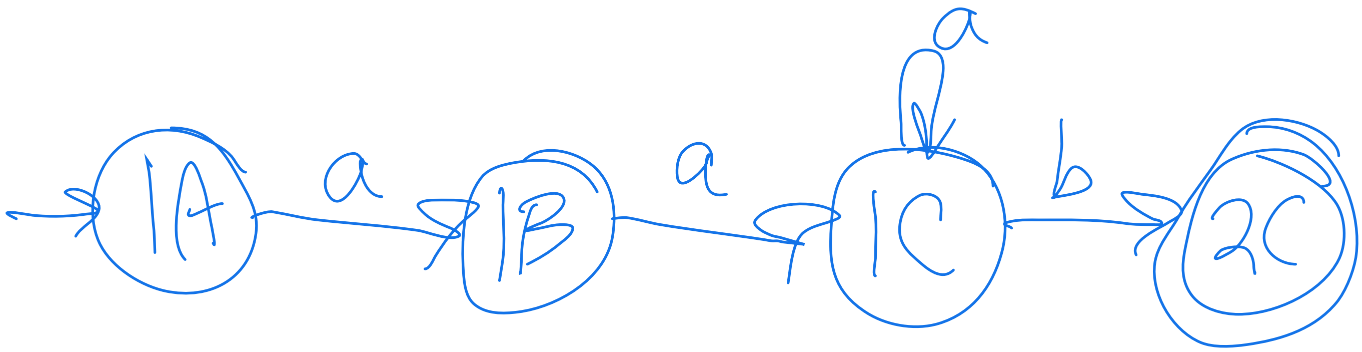
$F' = \{(q_i, p_j) \in Q' \mid q_i \in F_1 \text{ and } p_j \in F_2\}$

show $w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection

Example:





Regular languages are closed under

reversal	$\mathbf{L}^R$
difference	$\mathbf{L}_1 - \mathbf{L}_2$
right quotient	$\mathbf{L}_1/\mathbf{L}_2$
homomorphism	$\mathbf{h}(\mathbf{L})$

Right quotient

Def: $\mathbf{L}_1/\mathbf{L}_2 = \{x | xy \in \mathbf{L}_1$ **for some** $y \in \mathbf{L}_2\}$

Example:

$\mathbf{L}_1 = \{a^*b^* \cup b^*a^*\}$
$\mathbf{L}_2 = \{b^n | n$ **is even,** $n > 0\}$
$\mathbf{L}_1/\mathbf{L}_2 =$ a^*b^*

$aabb \in L_1/L_2$? yes

$\underbrace{aabb}_{L_1/L_2}\underbrace{bb}_{L_2}$

Theorem If L_1 and L_2 are regular, then L_1/L_2 is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

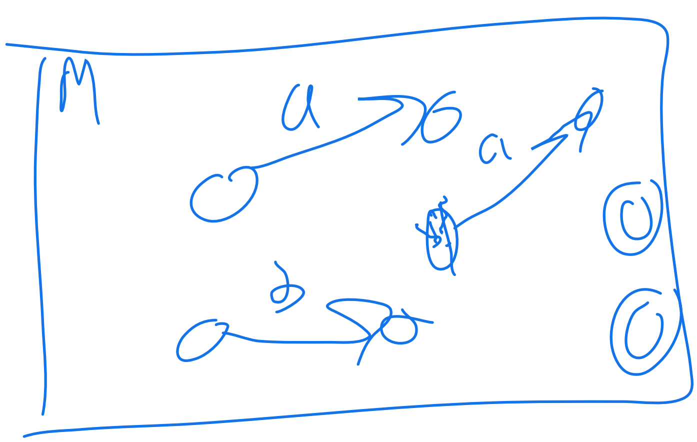
Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state i do

    Make i the start state (representing L_i')

if $L_i' \cap L_2 \neq \phi$
    put q_i in F' in M'

Changing start state to i



QED.

Homomorphism

Def. Let Σ, Γ be alphabets. A homomorphism is a function

$$\mathbf{h}\mathbf{:}\Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}\text{, } \Gamma = \{0, 1\}$$

$$\mathbf{h(a)=11}$$
$$\mathbf{h(b)=00}$$
$$\mathbf{h(c)=0}$$

$$\mathbf{h(bc)} = 000$$

$$\mathbf{h(ab^*)} = 11(00)^*$$

Questions about regular languages :

L is a regular language.

- **Given L, Σ, w$\in \Sigma^*$, is w$\in$L?**

  Construct a DFA and test if it accepts w

- **Is L empty?**

  $L = \{a^n b^m \mid n \geq 0, m \geq 0\} \cap \{b^n a^m \mid n > 1, m > 1\}$

  DFS

  Construct NFA $\Rightarrow$ DFA

  is there path from start state to final state

- **Is L infinite?**

  Construct DFA

  is there a cycle?

  DFS

  should know by n steps if n states

- **Does $L_1 = L_2$?**

  Construct $L_3 = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$

  If $L_3 = \phi$ then $L_1 = L_2$

Identifying Nonregular Languages

If a language L is finite, is L regular? yes

If L is infinite, is L regular? it depends

- $L_1 = \{a^n b^m | n > 0, m > 0\} =$ $aa^* bb^*$
- $L_2 = \{a^n b^n | n > 0\}$ NOT regular

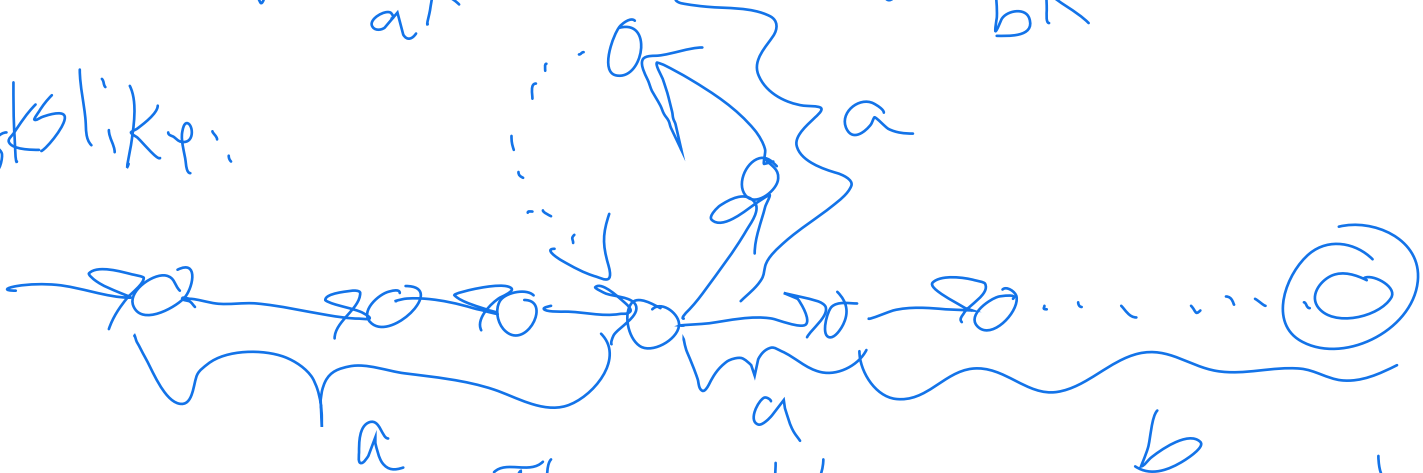
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- **Proof: Suppose L_2 is regular.**
  $\Rightarrow \exists$ DFA M that recognizes L_2

M has finite no. of states, k states

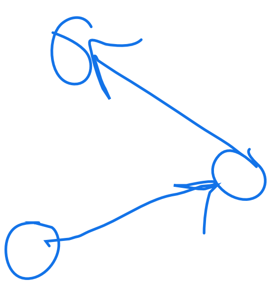
Consider a long string $a^k b^k \in L_2$

looks like:



There must be a cycle in the a's.

Go thru cycle one extra time. that string not in L



n states
cycle by n steps

L_2 can't be regular

there can't be a dfa ⟶ contradiction

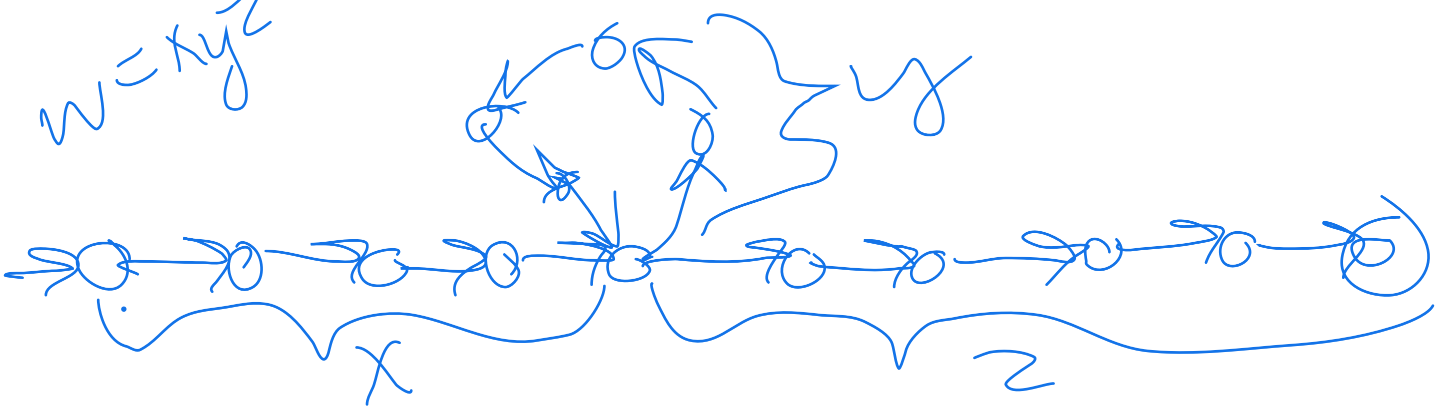
K from before

no of states in DFA if existed

Pumping Lemma: Let L be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^iz \in L \text{ for all } i \geq 0$$

$w = xyz$



To Use the Pumping Lemma to prove L is not regular:

- **Proof by Contradiction.**

  Assume L is regular.

  $\Rightarrow$ L satisfies the pumping lemma.

  Choose a long string w in L, $|w| \geq m$.

  Show that there is NO division of w into xyz (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in$L $\forall\ i \geq 0$.

  The pumping lemma does not hold. Contradiction!

  $\Rightarrow$ L is not regular. QED.

Example L=$\{a^n c b^n | n > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w =$ $a^m c b^m$

only way partition

$x = a^k \quad y = a^j \quad z = a^{m-k-j} c b^m$

should be true $xy^i z \in L \ \forall i$

pick $i=0$

$xy^0 z = xz = a^{m-j} c b^m \notin L$

Contradiction

$\Rightarrow L$ is not regular

$i=2$ also works

STOPPED HERE

Example L=$\{a^n b^{n+s} c^s | n, s > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose w= $a^m b^{2m} c^m = xyz$ $\quad a^n b^{m+1} c^{\text{any ok}}$

  So the partition is:

all possible partitions $\quad x = a^k \quad y = a^j, j > 0 \quad z = a^{m-k-j} b^{2m} c^m$

$\forall i \; xy^iz \in L$

$i = 0 \quad xz = a^{m-j} b^{2m} c^m \notin L \quad n(a) + n(c) \neq n(b)$

Contradiction!

$i = 2$ also works

L is not regular!

Example $\Sigma = \{a, b\}$**,**
L=$\{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ **the pumping lemma holds.**

  Choose w**=** $a^{m+1} b^m$

  So the partition is:

all the possible partitions partitioning of string

$x = a^k \quad y = a^j \quad z = a^{m+1-k-j} b^m$

$\forall i \; xy^iz \in L$

$i = 2 \quad xyyz = a^{m+1+j} b^m \in L$

not a contradiction

$i = 0 \quad xy^0z = xz = a^{m+1-j} b^m \notin L$

$j > 0 \quad n_a \not> n_b$

Contradiction!

L is not regular

Example L=$\{a^3b^nc^{n-3} | n > 3\}$

(shown in detail on handout)

L is not regular.

Prove several cases

aaabbb....bc........c

uvw

x y z

x y z

look at handout for details

stop here

To Use Closure Properties to prove L is not regular:

- **Proof Outline:**

  Assume L is regular.

  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

  closure properties $\Rightarrow$ L' is regular.

  Contradiction!

  L is not regular. QED.

Example L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a \;\; h(b) = a \;\; h(c) = b$
  $h(L) =$ $\{a^{n+3}b^{n-3} \mid n > 3\}$

$h(L)$ is regular

$\{bbbbbb\}$ is regular

$L' = h(L) \circ \{b^6\} = \{a^n b^n \mid n > 0\}$

~~$\{a^{n+3}b^{n+3} \mid n > 3\}$ is not regular~~

Contradiction!

$\Rightarrow$ L is not regular

$L'' = L' \cup \{ab, a^2b^2, a^3b^3, \ldots a^6b^6\}$

$L'' = \{a^nb^n \mid n > 0\}$ is not regular

Example L=$\{a^n b^m a^m | m \geq 0, n \geq 0\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.

$L1 = \{bb^* aa^*\}$

$L2 = L \cap L1 = \{b^n a^n | n > 0\}$

Define a homomorphism

$h(a) = b \quad h(b) = a$

$h(L2) = \{a^n b^n | n > 0\}$

not regular! Contradiction!

$\Rightarrow$ L is not regular

Example: $L_1 = \{a^n b^n a^n | n > 0\}$

L_1 **is not regular.**

Proof: Assume L_1 is regular

Let $L_2 = \{a^*\}$ L_2 is regular

$L_3 = L_1 \setminus L_2 = \{a^n b^n a^p | 0 \le p \le n, n > 0\}$

$L_4 = L_3 \cap \{aa^* bb^*\} = \{a^n b^n | n > 0\}$

L_4 is not regular

Contradiction

$\Rightarrow L_1$ is not regular