# Section: Other Models of Turing Machines

**Definition: Two automata are equivalent if they accept the same language.**

**Turing Machines with Stay Option**

**Modify $\delta$,** $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$

**Theorem Class of standard TM's is equivalent to class of TM's with stay option.**

**Proof:**

- **($\Rightarrow$): Given a standard TM M, then there exists a TM M' with stay option such that L(M)=L(M').**

easy

- (⇐): Given a TM M with stay option, construct a standard TM M' such that L(M)=L(M').

  M=(Q,$\Sigma, \Gamma, \delta, q_0$,B,F)

  M'=$(Q', \Sigma, \Gamma, \delta'', q_0', B, F')$

  For each transition in M with a move (L or R) put the transition in M'. So, for

$$\delta(q_i, a) = (q_j, b, \textbf{L or R})$$

  put into $\delta'$

$$\delta'(q_i', a) = (q_j', b, L \text{ or } R)$$

  For each transition in M with S (stay-option), move right and move left. So for

$$\delta(q_i, a) = (q_j, b, \mathbf{S})$$

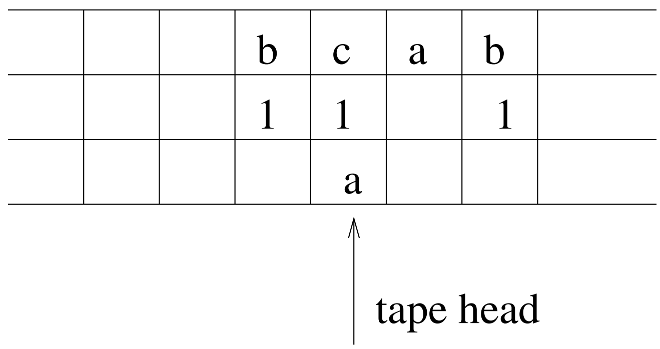
add a new state
add to $\delta'$

$$\delta'(q_i', a) = (q_{js}', b, R)$$
$$\delta(q_{js}', c) = (q_j', c, L) \quad \forall c \in \Gamma$$

  L(M)=L(M'). QED.

**Definition:** **A** *multiple track* **TM divides each cell of the tape into k cells, for some constant k.**

**A 3-track TM:**




**A multiple track TM starts with the input on the first track, all other tracks are blank.**

3-track

$\delta$**:** $Q \times \Gamma \times \Gamma \times \Gamma \to Q \times \Gamma \times \Gamma \times \Gamma \times \{L, R\}$

**Theorem Class of standard TM's is equivalent to class of TM's with multiple tracks.**

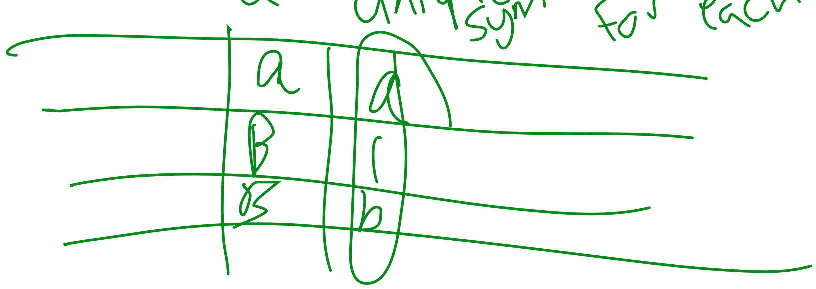
**Proof: (sketch)**

- **(⇒): Given standard TM M there exists a TM M' with multiple tracks such that L(M)=L(M').**

use first track

- **(⇐): Given a TM M with multiple tracks there exists a standard TM M' such that L(M)=L(M').**

a unique symbol for each triple




finite tape alphabet

⇒ finite unique symbols

Encode each comb. of 3 symbols w/ unique symbol

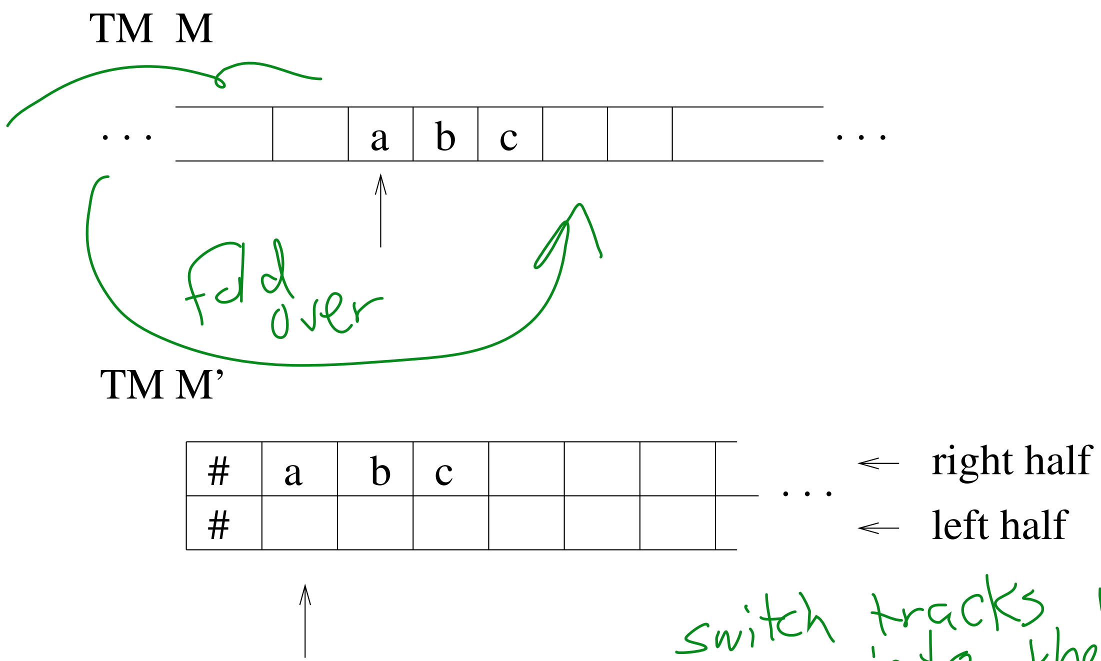
**Definition: A TM with a semi-infinite tape is a standard TM with a left boundary.**

**Theorem Class of standard TM's is equivalent to class of TM's with semi-infinite tapes.**

**Proof: (sketch)**

- **($\Rightarrow$): Given standard TM M there exists a TM M' with semi-infinite tape such that L(M)=L(M').**

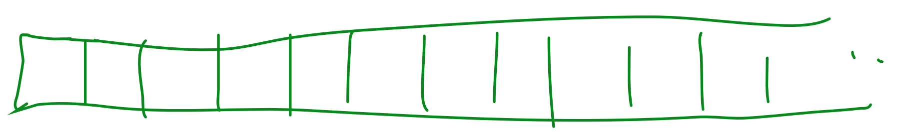
  **Given M, construct a 2-track semi-infinite TM M'**




switch tracks when run into the special marker at the end

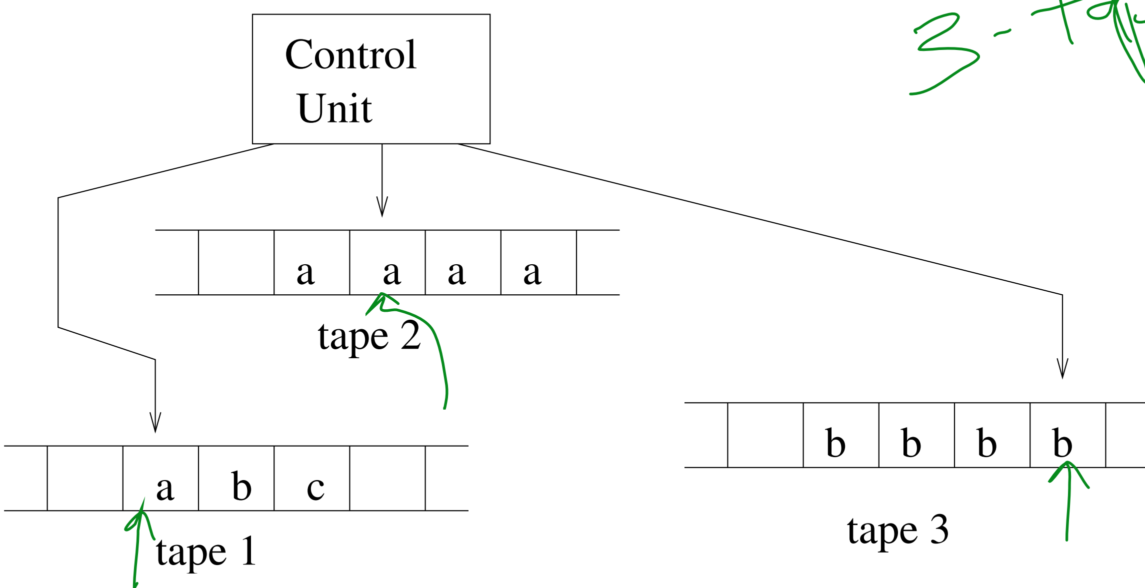
- **(⇐): Given a TM M with semi-infinite tape there exists a standard TM M' such that L(M)=L(M').**

should just work



**Definition: An Multitape Turing Machine is a standard TM with multiple (a finite number) read/write tapes.**

3-tape




3-tape

**For an n-tape TM, define $\delta$:**

$\delta: Q\times\Gamma\times\Gamma\times\Gamma \rightarrow Q\times\Gamma\times\Gamma\times\Gamma\times\{L,R\}\times\{L,R\}\times\{L,R\}$

**Theorem** Class of Multitape TM's is equivalent to class of standard TM's.

**Proof:** (sketch)

- ($\Leftarrow$): Given standard TM M, construct a multitape TM M' such that L(M)=L(M').

easy just run, just ignore other tapes

- ($\Rightarrow$): Given n-tape TM M construct a standard TM M' such that L(M)=L(M').

Create 2n track TM

3-tapes TM $\Rightarrow$ 6-track TM

marker

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | # | a | b | c | | | | |
| | | | # | 1 | | | | | | |
| | | | # | a | a | a | a | | | |
| | | | # | | 1 | | | | | |
| | | | # | b | b | b | b | | | |
| | | | # | | | | 1 | | | |

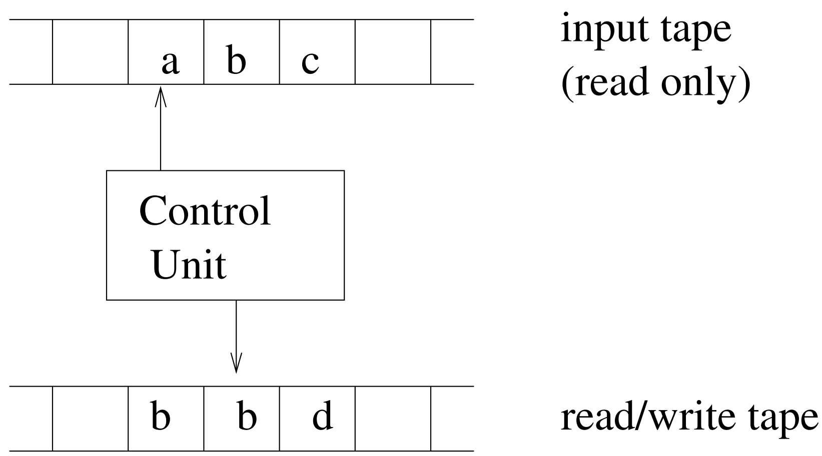
↑

one tape head

$\Rightarrow$ multtrack TM can be converted to standard TM.

**Definition: An Off-Line Turing Machine is a standard TM with 2 tapes: a read-only input tape and a read/write output tape.**

**Define $\delta$:** $Q \times \Sigma \times \Gamma \to Q \times \Gamma \times \{L, R\} \times \{L, R\}$



**Theorem** Class of standard TM's is equivalent to class of Off-line TM's.

similar to 2-track

**Proof:** (sketch)

- ($\Rightarrow$): Given standard TM M there exists an off-line TM M' such that L(M)=L(M').

- ($\Leftarrow$): Given an off-line TM M there exists a standard TM M' such that L(M)=L(M').

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | # | a | b | c | | | | |
| | | | # | 1 | | | | | | |
| | | | # | b | b | d | | | | |
| | | | # | | 1 | | | | | |

$\uparrow$

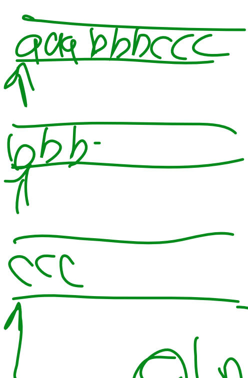
# Running Time of Turing Machines

## Example:

**Given L=$\{a^n b^n c^n | n > 0\}$. Given w$\in \Sigma^*$, is w in L?**

One tape was $\Theta(n^2)$

## Write a 3-tape TM for this problem.

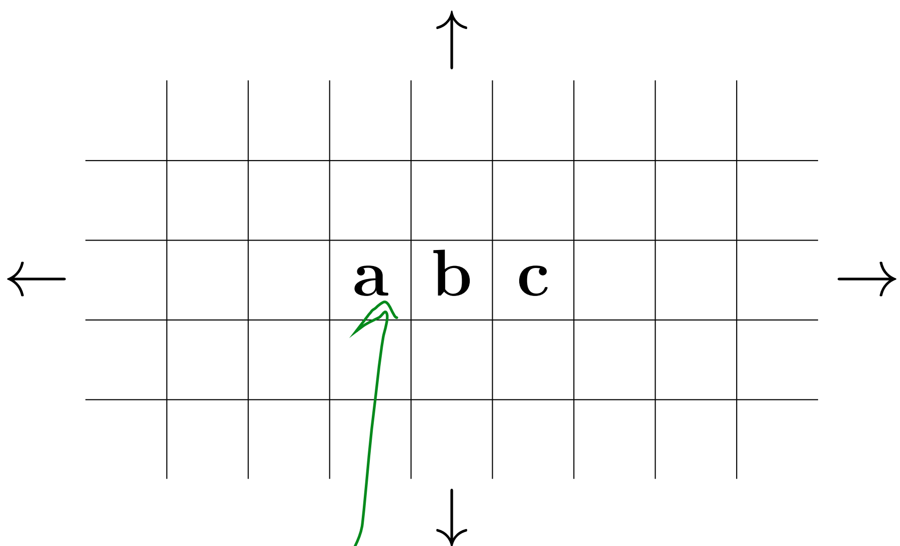



$\Theta(n)$

Copy B's to tape 2 $\Theta(n)$
Copy C's to tape 3
Start at beginning
Scan once & match a b, and a c for each a $\Theta(n)$

This model is more efficient! but not more powerful

**Definition: An Multidimensional-tape Turing Machine is a standard TM with a multidimensional tape**




Up
Down

**Define $\delta$:** $Q \times \Gamma \rightarrow Q \times \Gamma \times \{R, L, U, D\}$

**Theorem Class of standard TM's is equivalent to class of 2-dimensional-tape TM's.**

**Proof: (sketch)**

- **($\Rightarrow$): Given standard TM M, construct a 2-dim-tape TM M' such that L(M)=L(M').**

  easy, just run it

- **($\Leftarrow$): Given 2-dim tape TM M, construct a standard TM M' such that L(M)=L(M').**

−2,1 is 00#1

↑

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | **-1,2** | **1,2** | **2,2** | | | | |
| ← | **-2,1** | **-1,1** | **a 1,1** | **b 2,1** | **c 3,1** | | | |
| | **-2,-1** | **-1,-1** | **1,-1** | **2,-1** | | | | |
| | | | | | | | | |

↓

abc on tape

## Construct M’

and then convert 2-track TM to the standard TM.

add −1,1 # #0#1

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **#** | **a** | | | **#** | **b** | | | | **#** | **c** | | | | | |
| | | **#** | **1** | **#** | **1** | **#** | **1** | **1** | **#** | **1** | **#** | **1** | **1** | **1** | **#** | **1** | |

↑

unary 1's pos, 0's neg never use 0 to mean zero



for tape head {helpful to have one more track

**Definition: A *nondeterministic* Turing machine is a standard TM in which the range of the transition function is a set of possible transitions.**

**Define $\delta$:** $Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L,R\}}$

**Theorem Class of deterministic TM's is equivalent to class of nondeterministic TM's.**

**Proof: (sketch)**

- **($\Rightarrow$): Given deterministic TM M, construct a nondeterministic TM M' such that L(M)=L(M').**

  run it, no nondeterminism!

- **($\Leftarrow$): Given nondeterministic TM M, construct a deterministic TM M' such that L(M)=L(M').**

  **Construct M' to be a 2-dim tape TM.**

**A step consists of making one move for each of the current machines.**

**For example: Consider the following transition:**

$$\delta(q_0, a) = \{(q_1, b, R), (q_2, a, L), (q_1, c, R)\}$$

**Being in state $q_0$ with input abc.**

| | | | | |
|---|---|---|---|---|
| # | # | # | # | # |
| # | **a** | **b** | **c** | # |
| # | $q_0$ | | | # |
| # | # | # | # | # |

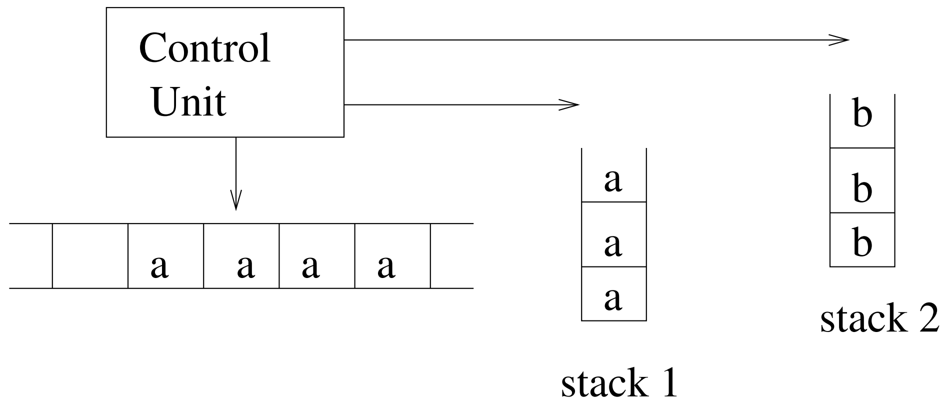
**The one move has three choices, so 2 additional machines are started.**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | # | # | # | # | # | # | | |
| | # | | **b** | **b** | **c** | # | | |
| | # | | | $q_1$ | | # | | |
| | # | | **a** | **b** | **c** | # | | |
| | # | $q_2$ | | | | # | | |
| | # | | **c** | **b** | **c** | # | | |
| | # | | | $q_1$ | | # | | |
| | # | # | # | # | # | # | | |
| | | | | | | | | |

2-dim TM →

to a standard TM,

**Definition: A 2-stack NPDA is an NPDA with 2 stacks.**




**Define $\delta$:** $Q \times \Sigma \times \Gamma \cup \{\lambda\} \times \Gamma \cup \{\lambda\} \rightarrow$ subsets of $Q \times \Gamma^* \times \Gamma^*$

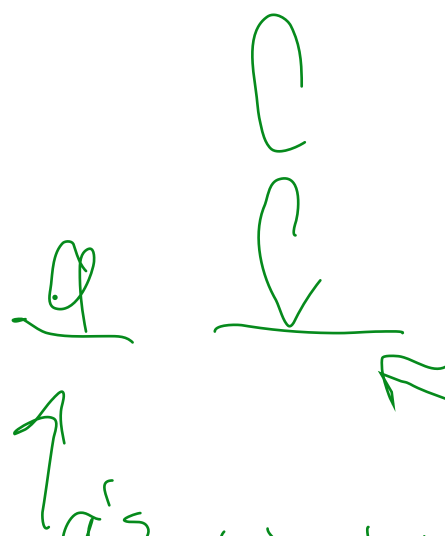
**Consider the following languages which could not be accepted by an NPDA.** Can you solve w/ 2-stack NPDA

1. **L**=$\{a^n b^n c^n | n > 0\}$ yes

2. **L**=$\{a^n b^n a^n b^n | n > 0\}$ yes

3. **L**=$\{w \in \Sigma^* |$ **number of** $a$**'s equals number of** $b$**'s equals number of** $c$**'s**$\}$, $\Sigma = \{a, b, c\}$ yes

accbaacbb

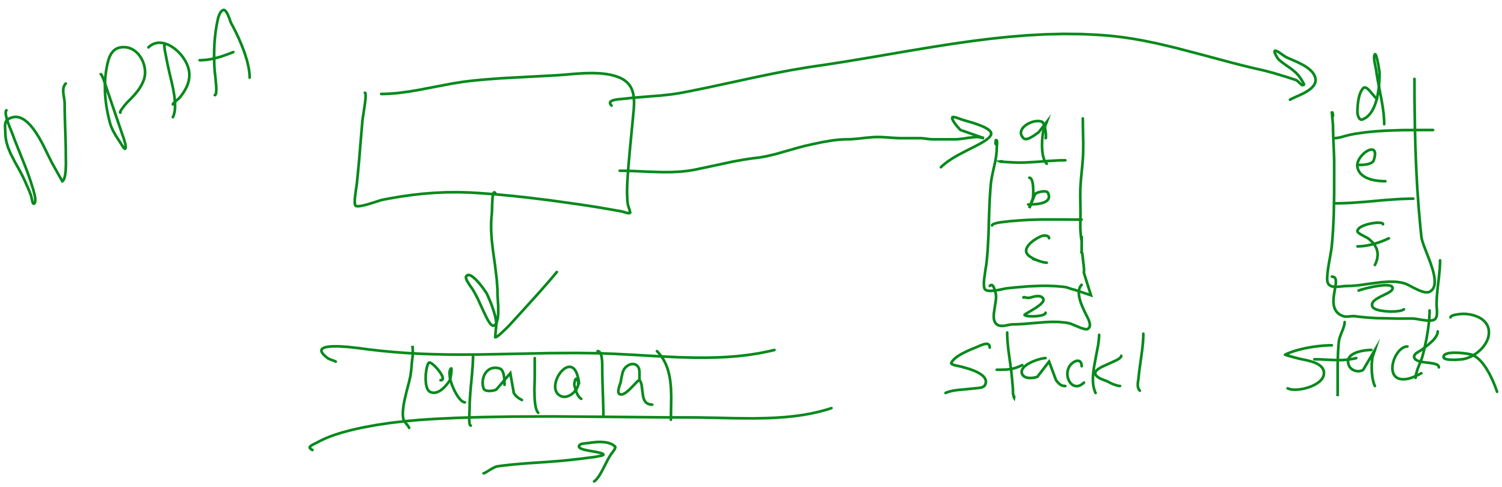


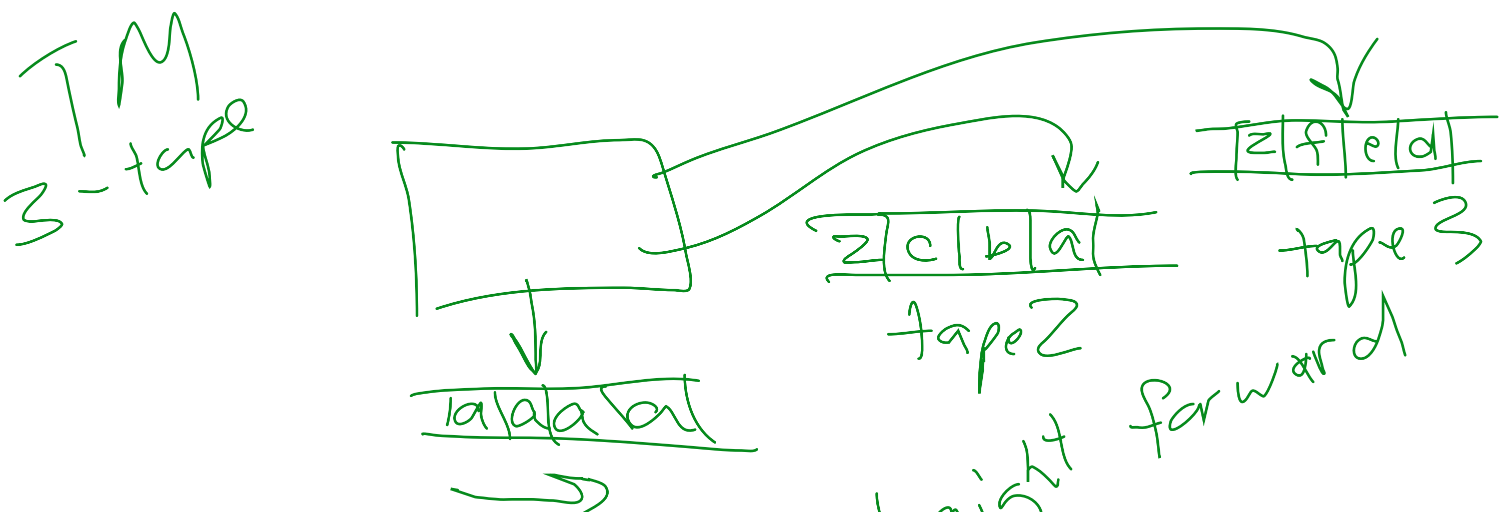

STOPPED HERE

**Theorem** Class of 2-stack NPDA's is equivalent to class of standard TM's.

**Proof:** (sketch)

- ($\Rightarrow$): Given 2-stack NPDA, construct a 3-tape TM M' such that L(M)=L(M').



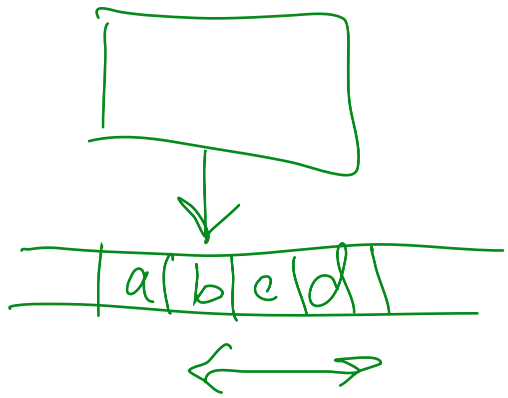




straight forward

$\Rightarrow$ standard TM

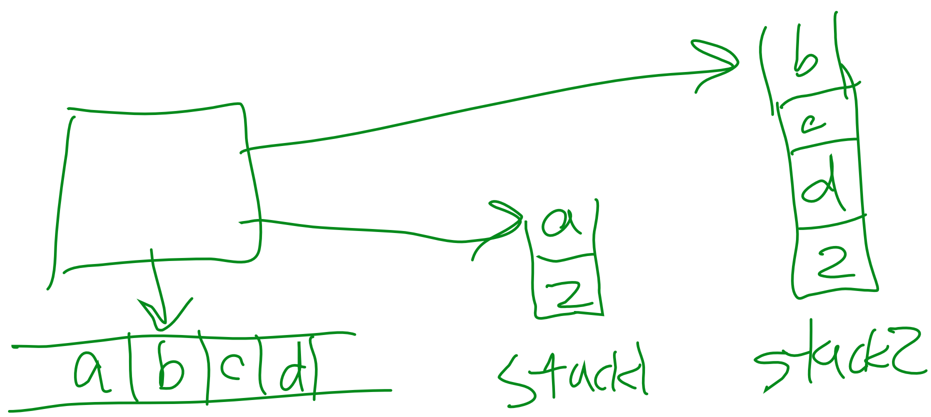
- **(⇐): Given standard TM M, construct a 2-stack NPDA M' such that L(M)=L(M').**

TM




NPDA

copy the input to stack




2-stack more powerful 1-stack

# Universal TM - a programmable TM

- **Input:**
  - **an encoded TM M**
  - **input string w**
- **Output:**
  - **Simulate M on w**

## An encoding of a TM

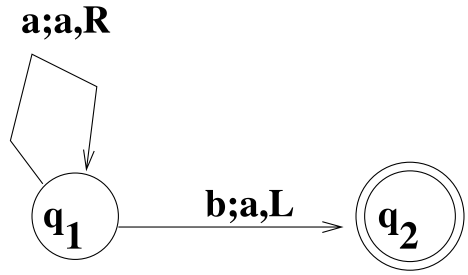
**Let TM M=$\{$Q,$\Sigma, \Gamma, \delta, q_1$,B,F$\}$**

- **Q=**$\{q_1, q_2, \ldots, q_n\}$
  **Designate $q_1$ as the start state.**
  **Designate $q_2$ as the only final state.**
  **$q_n$ will be encoded as n 1's**
- **Moves**
  **L will be encoded by 1**
  **R will be encoded by 11**
- $\Gamma = \{a_1, a_2, \ldots, a_m\}$
  **where $a_1$ will always represent the B.**

encode with 1's

$a_1$ 1
$a_2$ 11

For example, consider the simple TM:




$\Gamma$={B,a,b} which would be encoded as

The TM has 2 transitions,

$$\delta(q_1,\mathbf{a})=(q_1,\mathbf{a},\mathbf{R}), \quad \delta(q_1,\mathbf{b})=(q_2,\mathbf{a},\mathbf{L})$$

which can be represented as 5-tuples:

$$(q_1,\mathbf{a},q_1,\mathbf{a},\mathbf{R}),(q_1,\mathbf{b},q_2,\mathbf{a},\mathbf{L})$$

Thus, the encoding of the TM is:

**0101101011011010111011011010**

$q_1$ a $q_1$ a R $q_1$ b $q_2$ a L

**For example, the encoding of the TM above with input string "aba" would be encoded as:** TM

input string

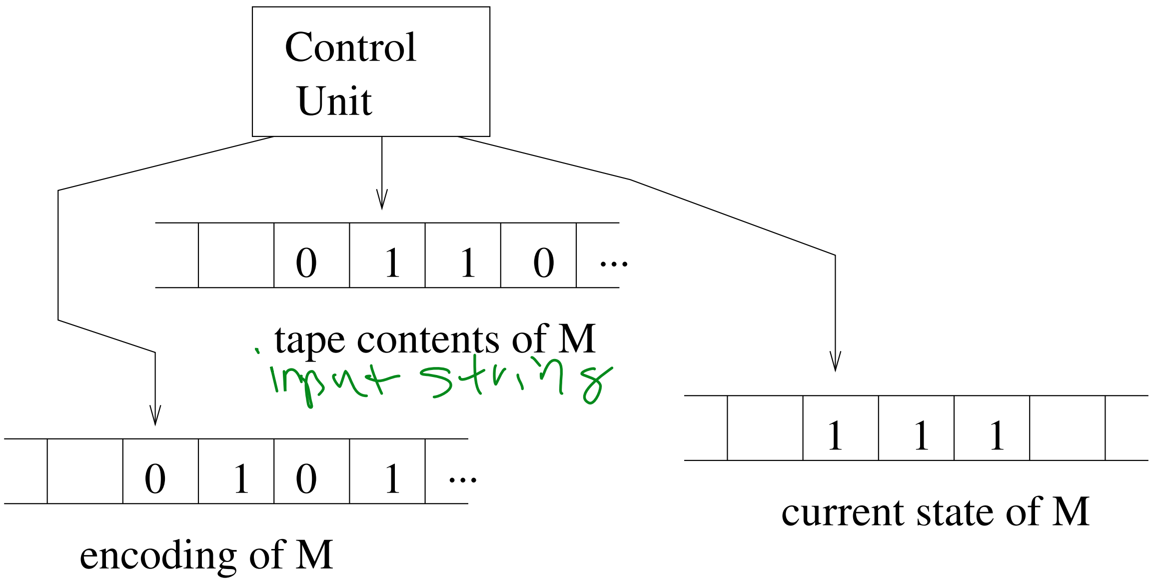
**010110101101101011101101101001101110110**

a b a

**Question: Given $w \in \{0,1\}^{+}$, is $w$ the encoding of a TM?**

# Universal TM

## The Universal TM (denoted $M_U$) is a 3-tape TM:



**Program for $M_U$**

1. **Start with all input (encoding of TM and string w) on tape 1. Verify that it contains the encoding of a TM.**
2. **Move input $w$ to tape 2**
3. **Initialize tape 3 to 1 (the initial state)**
4. **Repeat (simulate TM M)**
   (a) **consult tape 2 and 3, (suppose current symbol on tape 2 is $a$ and state on tape 3 is p)**
   (b) **lookup the move (transition) on tape 1, (suppose $\delta$(p,a)=(q,b,R).)**
   (c) **apply the move**
      - **write on tape 2 (write b)**
      - **move on tape 2 (move right)**
      - **write new state on tape 3 (write q)**

**Observation: Every TM can be encoded as string of 0's and 1's.**

⇒ unique no. for each TM

⇒ order all the TM by their number

**Enumeration procedure - process to list all elements of a set in ordered fashion.**

**Definition: An infinite set is *countable* if its elements have 1-1 correspondence with the positive integers.** 1, 2, 3, 4, 5,

**Examples:**

- **S = { positive odd integers }**

  1 2 3 4 5

  1, 3, 5, 7, 9

  yes
- **S = { real numbers }** no
- **S** $= \{w \in \Sigma^+\}$, $\Sigma = \{a, b\}$ yes a, b, aa, ab, ba, bb, in order
- **S = { TM's }** yes list all binary numbers in order, and check if TM → then list it
- **S = {(i,j) | i,j>0, are integers}**

  yes

| | | | | |
|---|---|---|---|---|
| (1,1), | (1,2), | (1,3), | (1,4) | (1,5),... |
| (2,1), | (2,2), | (2,3), | (2,4) | (2,5), |
| (3,1), | (3,2), | (3,3), | (3,4) | ... |
| (4,1), | (4,2), | (4,3), | (4,4), | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | |

## Linear Bounded Automata

**We place restrictions on the amount of tape we can use,**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | [ | **a** | **b** | **c** | ] | | | |
| | $\uparrow$ | | | | | | | |

**Definition: A linear bounded automaton (LBA) is a nondeterministic TM M=(Q,$\Sigma, \Gamma, \delta, q_0$,B,F) such that $[, ] \in \Sigma$ and the tape head cannot move out of the confines of []'s. Thus,**
$\delta(q_i, [) = (q_j, [, R)$**, and** $\delta(q_i, ]) = (q_j, ], L)$

**Definition: Let M be a LBA.**
**L(M)=**$\{w \in (\Sigma - \{[, ]\})^* | q_0[w] \overset{*}{\vdash} [x_1 q_f x_2]\}$

**Example: L=**$\{a^n b^n c^n | n > 0\}$ **is accepted by some LBA**

less powerful than TM!