Section: Properties of Context-free Languages

Which of the following languages are CFL?

- **L**=$\{a^n b^n c^j \mid 0 < n \leq j\}$
- **L**=$\{a^n b^j a^n b^j \mid n > 0, j > 0\}$
- **L**=$\{a^n b^j a^k b^p \mid n + j \leq k + p, n > 0, j > 0, k > 0, p > 0\}$
- **L**=$\{a^n b^j a^j b^n \mid n > 0, j > 0\}$

Pumping Lemma for Regular Language's: Let L be a regular language, Then there is a constant m such that $w \in L$, $|w| \geq m$, $w = xyz$ such that

- $|xy| \leq m$
- $|y| \geq 1$
- **for all** $i \geq 0$, $xy^i z \in L$

Pumping Lemma for CFL's **Let L be any infinite CFL. Then there is a constant m depending only on L, such that for every string w in L, with $|w| \geq m$, we may partition $w = uvxyz$ such that:**

$|vxy| \leq m$, **(limit on size of substring)**
$|vy| \geq 1$, **(v and y not both empty)**
For all $i \geq 0$, $uv^ixy^iz \in$L

- **Proof: (sketch) There is a CFG G s.t. L=L(G).**

  Consider the parse tree of a long string in L.

  For any long string, some nonterminal N must appear twice in the path.

Example: Consider
$L = \{a^n b^n c^n : n \geq 1\}$**. Show** L **is not a CFL.**

- **Proof: (by contradiction)**

  Assume L is a CFL and apply the pumping lemma.

  Let m **be the constant in the pumping lemma and consider** $w = a^m b^m c^m$**. Note** $|w| \geq m$**.**

  Show there is no division of w **into** $uvxyz$ **such that** $|vy| \geq 1$**,** $|vxy| \leq m$**, and** $uv^ixy^iz \in$**L for** $i = 0, 1, 2, \ldots$**.**

Thus, there is no breakdown of w into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, uv^ixy^iz is in L. Contradiction, thus, L is not a CFL. Q.E.D.

Example Why would we want to recognize a language of the type $\{a^n b^n c^n : n \geq 1\}$?

Example: Consider $L = \{a^n b^n c^p : p > n > 0\}$. Show L is not a CFL.

- **Proof: Assume L is a CFL and apply the pumping lemma. Let m be the constant in the pumping lemma and consider $w =$ ________________ Note $|w| \geq m$. Show there is no division of w into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in$L for $i = 0, 1, 2, \ldots$.**

Example: Consider $L = \{a^j b^k : k = j^2\}$. Show L is not a CFL.

- **Proof: Assume L is a CFL and apply the pumping lemma. Let m be the constant in the pumping lemma and consider**

  $w =$ ____________________

  Show there is no division of w into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in$L for $i = 0, 1, 2, \ldots$.

  Case 1: Neither v nor y can contain 2 or more distinct symbols. If v contains a's and b's, then $uv^2xy^2z \notin$L since there will be b's before a's.

  Thus, v and y can be only a's, and b's (not mixed).

Example: Consider
$L = \{w\bar{w}w : w \in \Sigma^*\}$**,** $\Sigma = \{a, b\}$**, where** $\bar{w}$ **is the string** w **with each occurence of** a **replaced by** b **and each occurence of** b **replaced by** a**. Show** L **is not a CFL.**

- **Proof: Assume L is a CFL and apply the pumping lemma. Let** m **be the constant in the pumping lemma and consider**

  $w =$ ________________

  Show there is no division of w **into** $uvxyz$ **such that** $|vy| \geq 1$, $|vxy| \leq m$**, and** $uv^ixy^iz \in$**L for** $i = 0, 1, 2, \ldots$**.**

Example: Consider $L = \{a^n b^p b^p a^n\}$**. L is a CFL. The pumping lemma should apply!**

Let $m \geq 4$ **be the constant in the pumping lemma. Consider** $w = a^m b^m b^m a^m$**.**

We can break w **into** $uvxyz$**, with:**

Chap 8.2 Closure Properties of CFL's

Theorem CFL's are closed under union, concatenation, and star-closure.

- **Proof:**

  Given 2 CFG $G_1 = (V_1, T_1, S_1, P_1)$ **and** $G_2 = (V_2, T_2, S_2, P_2)$

  - **Union:**

    Construct G_3 **s.t.** $\mathbf{L}(G_3) = \mathbf{L}(G_1) \cup \mathbf{L}(G_2)$.

    $G_3 = (V_3, T_3, S_3, P_3)$

– **Concatenation:**
Construct G_3 **s.t.** $\mathbf{L}(G_3) = \mathbf{L}(G_1) \circ \mathbf{L}(G_2)$**.**
$G_3 = (V_3, T_3, S_3, P_3)$

– **Star-Closure**
Construct G_3 **s.t.** $\mathbf{L}(G_3) = \mathbf{L}(G_1)^*$
$G_3 = (V_3, T_3, S_3, P_3)$

Theorem CFL's are NOT closed under intersection and complementation.

- **Proof:**
  - **Intersection:**

– **Complementation:**

Theorem: CFL's are closed under *regular* intersection. If L_1 is CFL and L_2 is regular, then $L_1 \cap L_2$ is CFL.

- **Proof: (sketch) We take a NPDA for L_1 and a DFA for L_2 and construct a NPDA for $L_1 \cap L_2$.**

  $M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_0, z, F_1)$ **is an NPDA such that L(M_1) = L_1.**

  $M_2 = (Q_2, \Sigma, \delta_2, q_0', F_2)$ **is a DFA such that L(M_2) = L_2.**

Example of replacing arcs (NOT a Proof!):

We must formally define δ_3. If

then

Must show

if and only if

Questions about CFL:

1. Decide if CFL is empty?

2. Decide if CFL is infinite?

Example: Consider $L = \{a^{2n}b^{2m}c^nd^m : n, m \geq 0\}$**. Show** L **is not a CFL.**

- **Proof: Assume L is a CFL and apply the pumping lemma. Let** m **be the constant in the pumping lemma and consider** $w = a^{2m}b^{2m}c^md^m$**.**

  Show there is no division of w **into** $uvxyz$ **such that** $|vy| \geq 1$, $|vxy| \leq m$**, and** $uv^ixy^iz \in$**L for** $i = 0, 1, 2, \ldots$**.**

  Case 1: Neither v **nor** y **can contain 2 or more distinct symbols. If** v **contains** a**'s and** b**'s, then** $uv^2xy^2z \notin$**L since there will be** b**'s before** a**'s.**

  Thus, v **and** y **can be only** a**'s,** b**'s,** c**'s, or** d**'s (not mixed).**

  Case 2: $v = a^{t_1}$**, then** $y = a^{t_2}$ **or** b^{t_3} $(|vxy| \leq m)$

  If $y = a^{t_2}$**, then**

$uv^2xy^2z = a^{2m+t_1+t_2}b^{2m}c^md^m \notin L$ **since** $t_1 + t_2 > 0$**, the number of** a**'s is not twice the number of** c**'s.**

If $y = b^{t_3}$**, then** $uv^2xy^2z = a^{2m+t_1}b^{2m+t_3}c^md^m \notin L$ **since** $t_1 + t_3 > 0$**, either the number of** a**'s (denoted n(**a**)) is not twice n(**c**) or n(**b**) is not twice n(**d**).**

Case 3: $v = b^{t_1}$**, then** $y = b^{t_2}$ **or** c^{t_3}

If $y = b^{t_2}$**, then** $uv^2xy^2z = a^{2m}b^{2m+t_1+t_2}c^md^m \notin L$ **since** $t_1 + t_2 > 0$**, n(**b**)**$> 2*$**n(**d**).**

If $y = c^{t_3}$**, then** $uv^2xy^2z = a^{2m}b^{2m+t_1}c^{m+t_3}d^m \notin L$ **since** $t_1 + t_3 > 0$**, either n(**b**)**$> 2*$**n(**d**) or** $2*$**n(**c**)>n(**a**).**

Case 4: $v = c^{t_1}$**, then** $y = c^{t_2}$ **or** d^{t_3}

If $y = c^{t_2}$**, then** $uv^2xy^2z = a^{2m}b^{2m}c^{m+t_1+t_2}d^m \notin L$ **since** $t_1 + t_2 > 0$**,** $2*$**n(**c**)>n(**a**).**

If $y = d^{t_3}$**, then**

$uv^2xy^2z = a^{2m}b^{2m}c^{m+t_1}d^{m+t_3} \notin L$ **since** $t_1 + t_3 > 0$**, either** $2*$**n(**c**)>n(**a**) or** $2*$**n(**d**)>n(**b**).**

Case 5: $v = d^{t_1}$**, then** $y = d^{t_2}$
then $uv^2xy^2z = a^{2m}b^{2m}c^m d^{m+t_1+t_2} \notin L$ **since** $t_1 + t_2 > 0$, $2*$**n(**d**)>n(**c**).**

Thus, there is no breakdown of w **into** $uvxyz$ **such that** $|vy| \geq 1$, $|vxy| \leq m$ **and for all** $i \geq 0$, uv^ixy^iz **is in** L**. Contradiction, thus,** L **is not a CFL. Q.E.D.**