Compsci 334 - Mathematical Foundations of CS
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Section: Properties of Regular Languages (Ch. 4) (handout)

Example

$L = \{a^n b a^n \mid n > 0\}$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in$ class
L_1 op $L_2 = L_3$
$\Rightarrow L_3 \in$ class

Example

L={x | x is a positive even integer}

L is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1 L_2$
$\bar{L}_1$
L_1^*

are regular languages.

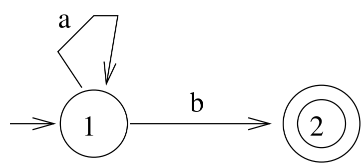
Proof(sketch)

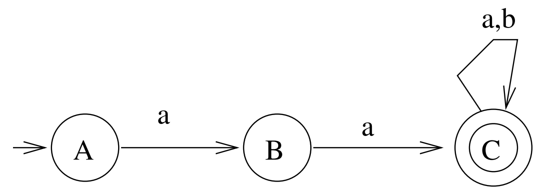
L_1 and L_2 are regular languages
$\Rightarrow \exists$ reg. expr. r_1 and r_2 s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
r_1^* is r.e. denoting L_1^*
$\Rightarrow$ closed under star-closure

complementation:
L_1 is reg. lang.
$\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$
Construct M' s.t.

intersection:
L_1 and L_2 are reg. lang.
$\Rightarrow \exists$ DFA M_1 and M_2 s.t.
$L_1 = L(M_1)$ and $L_2 = L(M_2)$
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
$Q' =$
δ':

Example:





Regular languages are closed under

reversal	L^R
difference	L_1-L_2
right quotient	L_1/L_2
homomorphism	h(L)

Right quotient

Def: $L_1/L_2 = \{x | xy \in L_1 \text{ for some } y \in L_2\}$

Example:

$L_1 = \{a^*b^* \cup b^*a^*\}$
$L_2 = \{b^n | n \text{ is even}, n > 0\}$
$L_1/L_2 =$

Theorem If L_1 and L_2 are regular, then L_1/L_2 is regular.

Proof (sketch)

$\exists$ DFA M=(Q,Σ,δ,q_0,F) s.t. L_1 = L(M).

Construct DFA M'=(Q,Σ,δ,q_0,F')

For each state i do
Make i the start state (representing $L_i^{'}$)

QED.

Homomorphism

Def. Let Σ, Γ be alphabets. A homomorphism is a function

$h: \Sigma \rightarrow \Gamma^*$

Example:

$\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$
h(a)=11
h(b)=00
h(c)=0

h(bc) =

h(ab*) =

Questions about regular languages :

L is a regular language.

- Given L, Σ, w$\in \Sigma^*$, is w$\in$L?

- Is L empty?

- Is L infinite?

- Does $L_1 = L_2$?

Ch. 4.3 - **Identifying Nonregular Languages**

If a language L is finite, is L regular?

If L is infinite, is L regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} =$
- $L_2 = \{a^n b^n | n > 0\}$

Prove that $L_2 = \{a^n b^n | n > 0\}$ **is ?**

- Proof: Suppose L_2 is regular.
  $\Rightarrow \exists$ DFA M that recognizes L_2

Pumping Lemma: Let L be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$\begin{array}{ll} |xy| \leq m & \\ |y| \geq 1 & \\ xy^i z \in L & \text{for all } i \geq 0 \end{array}$$

Meaning: Every long string in L (the constant m above corresponds to the finite number of states in M in the previous proof) can be partitioned into three parts such that the middle part can be "pumped" resulting in strings that must be in L.

To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.

  Assume L is regular.

  $\Rightarrow$ L satisfies the pumping lemma.

  Choose a long string w in L, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).

  Show that there is NO division of w into xyz (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^i z \in$L $\forall$ $i \geq 0$.

  The pumping lemma does not hold. Contradiction!

  $\Rightarrow$ L is not regular. QED.

Example L=$\{a^n c b^n | n > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w =$ where m is the constant in the pumping lemma. (Note that w must be choosen such that $|w| \geq m$.)

  The only way to partition w into three parts, $w = xyz$, is such that x contains 0 or more a's, y contains 1 or more a's, and z contains 0 or more a's concatenated with cb^m. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

  It should be true that $xy^i z \in$L for all $i \geq 0$.

Example L=$\{a^n b^{n+s} c^s | n, s > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose w=

  The only way to partition w into three parts, $w = xyz$, is such that x contains 0 or more a's, y contains 1 or more a's, and z contains 0 or more a's concatenated with the rest of the string $b^{m+s} c^s$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

Example $\Sigma = \{a, b\}$, L=$\{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose w=

  So the partition is:

Example L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof:**

  Assume L is regular. $\Rightarrow$ the pumping lemma holds.

  Choose $w = a^3b^mc^{m-3}$ where m is the constant in the pumping lemma. There are three ways to partition w into three parts, $w = xyz$. 1) y contains only a's 2) y contains only b's and 3) y contains a's and b's

  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide w into three parts s.t. the pumping lemma contraints were true).

  Case 1: (y contains only a's). Then x contains 0 to 2 a's, y contains 1 to 3 a's, and z contains 0 to 2 a's concatenated with the rest of the string b^mc^{m-3}, such that there are exactly 3 a's. So the partition is:

  $$x = a^k \quad y = a^j \quad z = a^{3-k-j}b^mc^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq 3$ for some constants k and j.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^2z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j-k}b^mc^{m-3}) = a^{3+j}b^mc^{m-3} \notin$ L since $j > 0$, there are too many a's. Contradiction!

  Case 2: (y contains only b's) Then x contains 3 a's followed by 0 or more b's, y contains 1 to $m - 3$ b's, and z contains 3 to $m - 3$ b's concatenated with the rest of the string c^{m-3}. So the partition is:

  $$x = a^3b^k \quad y = b^j \quad z = b^{m-k-j}c^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq m - 3$ for some constants k and j.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^0z = a^3b^{m-j}c^{m-3} \notin$ L since $j > 0$, there are too few b's. Contradiction!

  Case 3: (y contains a's and b's) Then x contains 0 to 2 a's, y contains 1 to 3 a's, and 1 to $m - 3$ b's, z contains 3 to $m - 1$ b's concatenated with the rest of the string c^{m-3}. So the partition is:

  $$x = a^{3-k} \quad y = a^kb^j \quad z = b^{m-j}c^{m-3}$$

  where $3 \geq k > 0$, and $m - 3 \geq j > 0$ for some constants k and j.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^2z = a^3b^ja^kb^mc^{m-3} \notin$ L since $j, k > 0$, there are b's before a's. Contradiction!

  $\Rightarrow$ There is no partition of w.

  $\Rightarrow$ L is not regular!. QED.

To Use Closure Properties to prove L is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**

  Assume L is regular.

  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

  closure properties $\Rightarrow$ L' is regular.

  Contradiction!

  L is not regular. QED.

Example L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof:** (proof by contradiction)

  Assume L is regular.

  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$

  $h(a) = a \quad h(b) = a \quad h(c) = b$

  $h(L) =$

Example L=$\{a^n b^m a^m | m \geq 0, n \geq 0\}$

L is not regular.

- **Proof:** (proof by contradiction)
  Assume L is regular.

Example: $L_1 = \{a^n b^n a^n | n > 0\}$

L_1 is not regular.

- **Proof**:
  Assume L_1 is regular.
  Goal is to try to construct $\{a^n b^n | n > 0\}$ which we know is not regular.
  Let $L_2 = \{a^*\}$. L_2 is regular.
  By closure under right quotient, $L_3 = L_1 \backslash L_2 = \{a^n b^n a^p | 0 \leq p \leq n, n > 0\}$ is regular.
  By closure under intersection, $L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n | n > 0\}$ is regular.
  Contradiction, already proved L_4 is not regular!
  Thus, L_1 is not regular. QED.