Analysis

1. **Review of the Brin-Page model for Page-Ranking (distribution).**

   Let $A \geq 0$ and $B \geq 0$ be of the same size and column stochastic. Let $C(\beta) = \beta A + (1-\beta)B \geq 0$ with the Bernoulli probability $\beta \in (0,1)$.

   (a) [M/T/F]
   Matrix $C(\beta)$ is column stochastic, with $\rho(C) = 1$. If both A and B are irreducible, then $C(\beta)$ is irreducible; and vice versa.
   Let $B = be^{\mathrm{T}}$ with $b > 0$ and $e^{\mathrm{T}}b = 1$. Then, B is irreducible, so is $C(\beta)$.

   (b) Let B be specified as in the previous problem. Let $x_p = x_p(b,\beta)$ be the Perron-Frobeneous vector of $C(\beta)$. Which one of the following is a correct description or representation of x_p, and make a correction if incorrect:

   $$C(\beta)x_p = x_p \tag{1}$$

   $$x_p = \lim_{k\to\infty} C^k(\beta)x_0, \quad x_0 > 0, \quad e^{\mathrm{T}}x_0 = 1. \tag{2}$$

   $$(I - \beta A)x_p = (1-\beta)b \tag{3}$$

   $$x_p = (1-\beta)\sum_{k=1}^{\infty} \beta^k A^k b \tag{4}$$

   (c) Find a way to prefix the distribution x_p.

   (d) Describe a simple iterative procedure in a finite number of steps to get an approximate $\widehat{x}_p$ to x_p.

   (e) Provided a solution $\widehat{x}_p$, suggest at least three criteria to assess the expected properties and accuracy.

2. [M/T/F]

   Use of the Neumann expansion for determining pair-wise reachability

   $$R_n = I + A + A^2 + \cdots + A^{n-1} \tag{5}$$

   That is, i can be reached by j if and only if $R_n(i,j) > 0$.

   Let $\alpha > 0$ be a scalar so that $\alpha\|A\| < 1$. Then $R(\alpha) = \sum_{k=0}^{\infty}(\alpha A)^k$ is well defined as the series converges. Then $R(\alpha)(i,j) > 0$ if and only if $R_n(i,j) > 0$.

   Furthermore, $R(\alpha)$ can be computed as the inverse of $I - \alpha A$.

   Remark: the inverse can be obtained by an LU factorization in $O(n^3)$ operations, the same order as one matrix-matrix product.

3. Provide a brief summary within 300 words on how to use a random-walk approach for node-to-vertex encoding and embedding, discussion with teammates, classmates and ChatGPT is encouraged.

4. **Review: what we know of the symmetric eigenvalue problem** (as the base for graph spectral analysis)

   Let A be an $n \times n$ real-valued symmetric matrix, $A^{\mathrm{H}} = A^{\mathrm{T}} = A$. Let $Ax_j = \lambda_j x_j$ be the eigen-pairs, $\lambda_j \in \mathbb{C}$ and $0 \neq x_j \in \mathbb{C}^{n\times 1}$.

   (a) [M/T/F] $\lambda_j \in \mathbb{R}$ because $x_j \mathrm{H} A x_j = \lambda_j x_j^{\mathrm{H}} x_j$
   One can index the eigenvalues in non-descending order, $\lambda_{\min} = \lambda_1 \leq \lambda_j \leq \lambda_n = \lambda_{\max}$.

   (b) [M/T/F] $A\mathrm{real}(x_j) = \lambda_j \mathrm{real}(x_j)$ and $A\mathrm{imag}(x_j) = \lambda_j \mathrm{imag}(x_j)$ therefore, the eigenvectors of A can be made real-valued.

   (c) Prove or disprove the following statement: If $\lambda_i \neq \lambda_j$, then $x_j^{\mathrm{T}} x_j = 0$.

   (d) [M/T/F] If the dimension of the invariant subspace of A associated with λ_j is greater than 1. Assume Q_j be the set of orthonormal vectors spanning the subspace, then $AQ_j = \lambda_j Q_j$.

   (e) Optional. [M/T/F] For every eigenvalue λ_j, its geometric multiplicity is equal to its algebraic multiplicity. This implies that A has a complete eigenvector system.

   (f) [M/T/F] An EVD of A can be expressed as follows, $A = Q\Lambda Q^{\mathrm{T}}$ where Λ diagonal and real-valued and Q is orthogonal.

   (g) [M/T/F] $\|A\|_F^2 = \sum_{ij} A^2(i,j) = \sum_j \lambda_j^2$ and $\|A\|_2 = \max\{\ |\lambda_{\max}|, |\lambda_{\min}|\ \}$, Consequently, $\|A\|_2 = \lambda_{\max}$ if and only if A is semi-positive definite.

   (h) [M/T/F] Assume in addition that $A \geq 0$ elementwise, with $d = Ae > 0$. Then the random-walk matrix $A_{\mathrm{w}} = AD^{-1}$ is symmetric if and only of d is constant. Nonetheless, $\lambda_j(A_w) \in \mathbb{R}$.

5. **The normalized graph Laplacians to edge weighted graphs: spectral structures & applications**

   Let $G(V, E, E)$ be a graph with non-negative edge weights, where W is the weight function, $W : E \to \mathbb{R}_+$. Let A be the adjacency matrix (with edge weights). Let B be the incidence matrix without edge weights. Define

   $$B_w \triangleq BD_e^{1/2}, \qquad L_w \triangleq B_w B_w^{\mathrm{T}}, \quad D_e = \mathrm{diag}(W). \tag{6}$$

   All the Laplacians, above or below, are defined as the gram product of a weighted incidence matrix.

   (a) [M/T/F]
   For any x, $x^{\mathrm{T}} L_w x \geq 0$, i.e., L_w is semi-positive definite, and $L_w e = 0$, i.e., L_w has zero eigenvalue(s), not positive definite.

   (b) [M/T/F]
   $L_w = D - A$, where A is the weighted adjacency matrix and $D = \mathrm{diag}(Ae)$.

   (c) [M/T/F]
   Graph G is connected if and only if the Fiedler value is positive. Consequently, the Fiedler value of L (unweighted) is nonzero if and only if the Fiedler value of L_w is nonzero0.

(d) Let $B_w = BD_e^{1/2}$. Describe the vertex scaling matrix D_v so that $\widehat{B}_w = D_v^{-1/2} B D_e^{1/2}$ is normalized in rows.

(e) [M/T/F]
Let $\widehat{L}_w = \widehat{B}_w \widehat{B}_w^{\mathrm{T}}$. Then, $\widehat{L}_w = I - \widehat{A}$, where $\widehat{A} = D_v^{-1/2} A D_v^{-1/2}$.
Furthermore, $\lambda_j(\widehat{A})$ is real and within $[-1.1]$.

(f) [M/T/F]
Let B_+ be the incidence matrix with $B(:,\ell) = e_i + e_j$ for $\ell = (i,j) \in E$. Let $B_{+,w} = BD_e^{1/2}$. Then, $\widehat{B}_{+,w} = D_v^{-1/2} B D_e^{1/2}$ is normalized in rows by the same vertex scaling. Then, $\widehat{L}_{+,w} = I + \widehat{A}$, where $\widehat{L}_{+,w}$ is the Laplacian as the gram product of $\widehat{B}_{+,w}$.

(g) Give a brief interpretation of element $\widehat{L}_+(i,j)$ in terms of neighborhood similarities.

(h) Verify (in brief expressions) the following equalities and inequalities

$$\begin{aligned} \lambda_j(\widehat{A}) &\in [-1.1], \quad j = 1:n \\ \widehat{A} &= Q\,\widehat{\Lambda}\,Q^{\mathrm{T}}, \quad \Lambda = \mathrm{diag}(\lambda_j), \quad Q^{\mathrm{T}}Q = I_n \\ \widehat{L}_w &= Q(I - \widehat{\Lambda})Q^{\mathrm{T}} \\ \widehat{L}_{+,w} &= Q(I + \widehat{\Lambda})Q^{\mathrm{T}} \end{aligned} \tag{7}$$

(i) [M/T/F]
Let G be connected. Let $d = Ae$ be the degree vector. Then, $d^{1/2}$ is the null eigenvector of $\widehat{L}_w$ and the Perron vector of $\widehat{L}_{+,w}$. The Fiedler vector of $\widehat{L}$ is the eigenvector associated with the second largest eigenvalue of $\widehat{L}_{+,w}$.
In any spectral approximation of graph G to preserve the neighborhood similarity, it is necessary to preserve at least the two principle eigenvectors of $\widehat{L}_{+,w}$.

6. In data-driven, evidence-based research, a frequent issue is to identify a random phenomenon and/or the deviation from it. List at least three types of random graphs.

The following are to initiate more of mental and analytical exercise for class projects.

7. Optional. Describe briefly the (matching) model used in `isoMap` for mapping graph nodes to vectors in a metric space.

8. Optional. Describe briefly the (matching) model used in `t-SNE` for a point cloud in a high-dimensional space to a point cloud in 2D/3D space. Then use this model to construct a graph.

9. Optional. Describe briefly approach extending the analysis of static graphs to time-varying graphs.

10. Optional. Describe briefly how to measure the similarity between two neighborhood in a digraph and how to extend the Laplacian spectral analysis of an undirected graph to a digraph.