1 Discrete Laplacian

Assume for the time being that a graph $G(V, E)$ under consideration is undirected. Let $n = |V|$ and $m = |E|$.

Connectivity

- ☐ The incidence matrix B: vertices vs edges

$$B(:, \ell) = \pm(e_i - e_j), \quad \ell = (i, j) \in E \tag{1}$$

By the construction, $e^{\mathrm{T}} B = 0$, e is a left singular vector of B associated with the zero singular value.

Or,

$$B_+(:, \ell) = e_i + e_j, \quad \ell = (i, j) \in E \tag{2}$$

By the construction, $e^{\mathrm{T}}(1:n)B = 2e^{\mathrm{T}}(1:m)$.

We have $|B| = |B_+|$ and that the degrees are the row sums: $d = |B|e(1:m)$.

- ☐ Adjacency matrix A: vertices vs. vertices

$$A(i, j) > 0 \iff (i, j) \in E \tag{3}$$

The (plain) Laplacian L: vertices vs. vertices

$$L = BB^{\mathrm{T}} = \mathrm{diag}(d) - A \tag{4}$$

By construction, L is symmetric, semi-positive definite, and $Le = 0$, i.e., e is an eigenvector associated with the minimal eigenvalue 0.

Decomposition and aggregation of L in terms of local Laplacian:

- ☐ Edge Laplacian

$$\begin{aligned} L(A) &= BB^{\mathrm{T}} \\ &= \sum_{\ell \in E} B(:, \ell) B(:, \ell)^{\mathrm{T}} = \sum_{\ell \in E} L(\ell) \\ &= \sum_{(i,j) \in E} (e_i - e_j)(e_i - e_j)^{\mathrm{T}} \qquad \text{\% outer product terms} \\ &= \sum_{(i,j) \in E} (\, A(i, j) I_2 - A([i, j], [i.j]) \,) \end{aligned} \tag{5}$$

- ☐ Laplacians of the neighborhood star graphs

$$\begin{aligned} L(G) &= \frac{1}{2} \sum_{i \in V} \sum_{j \in \mathcal{N}(i)} (e_i - e_j)(e_i - e_j)^{\mathrm{T}} \\ &= \frac{1}{2} \sum_{i \in V} L(S_i), \end{aligned} \tag{6}$$

where S_i is the star graph centered at vertex i/

1.1 Laplacians as metric kernels

Every Laplacian is semi-positive definite. It is positive when restricted to the space of $P_e\mathbb{R}^{\mathrm{n}}$, where the vertex functions/signals/vectors are of zero mean, $P_e = I - ee^{\mathrm{T}}/n$ is the projector into the zero-mean vector space.

The Laplacian introduces a metric, an inner product (a bilinear function) in the vector space $P_e\mathbb{R}^{\mathrm{n}}$,

$$< x, y >_L = x^{\mathrm{T}} L y, \quad x, y \in P_e\mathbb{R}^{\mathrm{n}}.$$

That is, the Laplacian serves as the kernel matrix of the metric.

Consequently, we have

- ◇ A measure of the vector length,

$$\|x\|_L^2 =< x, x >= x^{\mathrm{T}} L x$$

- ◇ A measure of the (squared) distance between two vectors

$$\|x - y\|_L^2 =< x - y, x - y >_L = (x - y)^{\mathrm{T}} L (x - y)$$

- ◇ A measure of the angle θ between two nonzero vectors x and y in $P_e\mathbb{R}^{\mathrm{n}}$,

$$\cos(\theta) = \frac{< x, y >_L}{\|x\|_L \cdot \|y\|_L}.$$

- ◇ The Rayleigh quotient

$$\frac{x^{\mathrm{T}} L x}{x^{\mathrm{T}} x}$$

  is the ratio between the squared length of x by the metric with kernel L and the squared length of x by the metric with kernel I.

Local metrics by local Laplacians

The 'curved' stories are with the local Laplacians. Let $G_i = G(V_i.E_i)$ with $V_i = \mathcal{N}[i]$ and $E_i = E(\mathcal{V}_\rangle \times \mathcal{V}_\rangle)$. The adjacency matrix is $A_i = A(V_i, V_i)$.

- ▷ With a local Laplacian $L(G_i)$, the metric

$$< x_i, y_i >_{L_i} = x_i^{\mathrm{T}} L_i y_i \tag{7}$$

  is local to the vector space associated with the neighborhood of vertex i, $V_i = \mathcal{N}[i]$. Specifically, the vector space associated with V_i is the space of vertex functions or signals, i.e., vectors, with their vertex supports restricted to V_i.

- ▷ In relation to the metric induced by Laplacian $L(G)$.

$$< x_i, y_i >_{L_i} = x_i^{\mathrm{T}} L_i y_i = \bar{x}_i^{\mathrm{T}} L \bar{y}_i \tag{8}$$

  where $\bar{x}_i$ pads x_i with zeros. In terms of the minimal or maximal Rayleigh quotient values,

- ▷ Minimal deviation: Local connectivity and the global connectivity

$$\lambda_2(G) = \min_{e^{\mathrm{T}} x = 0} \frac{x^{\mathrm{T}} L x}{x^{\mathrm{T}} x} \le \lambda_2(G_i) = \min_{e^{\mathrm{T}} x_i = 0} \frac{x_i^{\mathrm{T}} L_i x_i}{x_i^{\mathrm{T}} x_i} \tag{9}$$

- ▷ Maximum deviation:

$$\lambda_{\max}(G_i) = \max_{e^{\mathrm{T}} x_i = 0} \frac{x_i^{\mathrm{T}} L_i x_i}{x_i^{\mathrm{T}} x_i} \le \lambda_{\max}(G_i) = \max_{e^{\mathrm{T}} x = 0} \frac{x^{\mathrm{T}} L x}{x^{\mathrm{T}} x} \tag{10}$$

- ▷ Example: grid graphs.

1.2 Local geometric weights

XS: *in progress ...*

1.3 Translation on discrete graphs

XS: *in progress ...*