Midterm II solutions

1. Let us assume \( n = 2^k \). For \( 2^0, 2^1, 2^2, \ldots, 2^k \)th operations it will cost us the corresponding power of 2. For the rest \( n - \lg n \) operations its a unit cost. Hence adding all the costs and dividing by \( n \),

\[
\frac{\left( \sum_{i=0}^{k} 2^i + n - \lg n \right)}{n} \\
= \frac{(2^{k+1} - 1 + n - \lg n)}{n} \\
= \frac{(2n - 1 + n + \lg n)}{n} \\
= O(1)
\]

2. An undirected connected graph will have no cycle if and only if it is a tree. That is, \(|V| = |E| + 1\). Any graph having more than \(|V| - 1\) edges will have a cycle. So the most simple algorithm will be to go through the adjacency list counting the number of edges each vertex is connected to. Since each edge will be in the list twice, once we reach a count of \(2(|V| - 1) + 1\) edges we know the graph has a cycle and stop the traversing there. Else if at the end of the list we have \(2(|V| - 1)\) edges, we know there is no cycle.

3. (a) They will be identical. For Kruskal, we sort edges and pick the smallest ones. If we sort the edges for both graphs we get the same order of the corresponding edges. Thus Kruskal will pick the same edges for each graph.

(b) Counter Example: Consider a graph with 2 nodes and each node attached to the other two. \( V = a, b, c \) and \( w(a, b) = 2, w(b, c) = 3, w(a, c) = 4 \). Thus the MST will contain edges \((a, b)\) and \((b, c)\). So the shortest path from \(a\) to \(c\) in the tree will be of cost 5, but the shortest path is \((a, c) = 4\).

4. (a) It is the same as Dijkstra \(O(E + V \lg V)\).

(b) Remove each edge from the cheapest distribution as found from the first part and recompute the distribution cost by applying Dijkstra. Find the maximum difference between these costs and the original one. That is the answer. \(O(V(E + V \lg V))\).

5. The total flow is 14. I am not showing the steps, but the final flow will be following:

A→B = 3  
A→D = 7  
A→C = 4  
B→E = 3  
D→E = 4  
D→F = 3  
C→F = 4  
F→E = 1  
E→G = 8  
F→G = 6

The cut : A,B,C,D,F and E,G