Part I.

1. (15) Based on numerical experiments with the **Runge Example** in Part II (which was first given in 1901 by Runge), answer the following questions.
   (a) The complexity for each interpolation method used (including the evaluation method).
   (b) The accuracy for each interpolation method at non-nodal points.
   (c) The interpolating polynomial is oscillating while the Runge function is not [y/n?], why ?
   (d) The discrepancy between the Runge function $f(x)$ and the interpolating polynomial $p_n(x)$ is unbounded as $n$ increases [y/n?], why ?
   (e) Are these experiments supporting or contradictory to Weierstrass Approximation theorem ? Give some brief comments.

2. Verify that the following are inner products are well defined in their respective vector spaces.
   (a) (5) On $V = C[a, b]$ over field $\mathcal{R}$,
   $$\langle u, v\rangle = \int_a^b u(t)v(t)dt$$
   (b) (5) On $V = C^n$ over field $\mathcal{C}$,
   $$\langle u, v\rangle = u^{\mathrm{H}}v = \sum_{i=1}^{n} \bar{u}_i v_i$$

3. Apply Schwarz inequality to verify the following general results.
   (a) (5)
   $$\|u + v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 \quad \text{iff} \quad \langle u, v\rangle = 0.$$
   (b) (8) If $\|u\|_2\|v\|_2 \neq 0$, then
   $$-1 \leq \frac{\langle u, v\rangle}{\|u\|_2\|v\|_2} \leq 1.$$
   Let $V$ be a vector space over the real field. Give some reason(s) why the angle between two non-zero vectors in $V$ is defined in terms of COS instead of SIN
   $$\cos(\theta(u, v)) = \langle \frac{u}{\|u\|_2}, \frac{v}{\|v\|_2}\rangle.$$

4. Apply Schwarz inequality to verify the following specific claims.

   (a) (8)

   $$(\sum_{i=0}^{N} \alpha_i \beta_i)^2 \leq (\sum_{i=0}^{N} \alpha_i^2)(\sum_{i=0}^{N} \beta_i^2), \quad \alpha_i, \beta_i \in \mathcal{R}.$$

   Specify when the equality holds true. In a particular case,

   $$\sum_{k=0}^{N} \frac{x^k}{\sqrt{k!}} < e^{x/(2\alpha)} \sqrt{\frac{1-(\alpha x)^{N+1}}{1-\alpha x}}, \quad x \in [0, 2],$$

   where $\alpha$ is any number such that $\alpha x \neq 1$. Explain why the bound on the right can not be reached.

   (b) (8)

   $$(\int_a^b f(x)g(x)dx)^2 \leq (\int_a^b f^2(x)dx)(\int_a^b g^2(x)dx), \quad f, g \in C[a, b].$$

   Specify when the equality holds true. In a particular case,

   $$\int_{\pi/6}^{\pi/2} \sqrt{\frac{\cos(t)}{t}} dt < 1$$

   Explain why the bound on the right can not be reached.

5. (10) More understanding of the PageRank paper.

Part II. MATLAB EXPERIMENTS.

1. (24) Runge Example on polynomial interpolation.

   Runge's function is defined as follows

$$\mathbf{runge}(x) = \frac{1}{1+x^2}, \quad x \in [-5, 5].$$

   Test the following interpolation methods with Runge function and plot the interpolating functions for the following cases :

   (a) Distribution of interpolation nodes : equally spaced, non-equally spaced

   (b) Interpolation over the whole interval :
   Lagrange approach with Neville's evaluation method
   Vandermonde approach with Horner's evaluation method

   (c) Integrated interpolations over sub-intervals :
   Linear splines
   Quadratic splines
   Cubic splines

   (d) **optional.** (4) Another interpolation method.

2. (12) Parametric interpolation and Typeface design (Calligraphic letter drawing).

   Provide MATLAB functions/scripts for typeface setting with Bézier curves. Use the following example.

| $i$ | $x_i$ | $y_i$ | $\alpha_{i-1}$(end) | $\beta_{i-1}$(end) | $\alpha_i$(begin) | $\beta_i$(begin) |
|---|---|---|---|---|---|---|
| 0 | 3 | 6 | | | 3.3 | 6.5 |
| 1 | 2 | 2 | 2.8 | 3.0 | 2.5 | 2.5 |
| 2 | 6 | 6 | 5.8 | 5.0 | 5.0 | 5.8 |
| 3 | 5 | 2 | 5.5 | 2.2 | 4.5 | 2.5 |
| 4 | 6.5 | 3 | 6.4 | 2.8 | | |

   **Optional.** (6) Provide an interface for user interaction with Calligraphic letter drawing.