Part I.

1. (i) (6) Find out the precision degree of the following quadratures

   (a) the rectangle rules : the left rectangle rule, the right rectangle rule, the midpoint rule

   (b) the Trapezoidal rule

   (c) the Simpson's rule

   (ii) (6) For any given function f that is sufficiently smooth over an integration domain $[a, b]$, find out the dominant terms (including the orders and the associated coefficients) in the local and global errors in the numerical values obtained by each of the above quadratures (you may tabulate your results).

2. Partition refinement and quadrature composition.

   Let Q_1 be a quadrature of precision degree d. Then the local error for $(x - m)^d$ over Δ is $E((x - m)^d, \Delta) = c \cdot h^{d+1}$, for some constant c, where $h = |\Delta|$. Let Q_2 be the quadrature of Q_1 applied to the two equal-space subintervals of Δ.

   (a) (8) Derive the integration error in Q_2 for $(x - m)^{d+1}$ over Δ. Apply the general result to the specific quadrature rules in the previous problem.

   (b) (10) Compose a new quadrature Q from Q_1 and Q_2 such that Q is of precision degree $(d + 1)$.

3. (10) Based on the above study, design an adaptive algorithm for automatic error estimation and partition refinement, based on a quadrature of degree d, provided with

   (a) a bound on the global error over the entire integration domain,

   (b) a bound on the maximum number of sampling points,

   (c) an initial partition of the integration domain, and

   (d) a subroutine for evaluation of the integrand that is sufficiently smooth.

4. (10) Write a brief summary for each problem and its solution in the implementation part.

Part II.

1. (15) Numerical computation of definite integrals and their applications. Provide a subroutine for adaptive integration with Simpson's rule. Use the subroutine to the following problems.

   (a) Problem 9 in Exercise set 4.6. Plot the results in t-c plane, t-s plane and c-s plane with $t_j = j * h$ and $h = 1/n$ for $n = 5, 25, 50$.

   (b) Compute the normal distribution function $\text{erf}(x)$ with $x = j * h$, $h = 0.25$ and $j = 1 : 20$. The error tolerance at $x = 5$ is $\tau(5) = 10^{-5}$.

   OPTIONAL. (5) Provide a visualization of the adaptive partition.

2. (25) Numerical simulation of triggering propagation in 2-D cells (see the separate description and provided codelet).

3. (10) Provide a MATLAB function to compute a segment of the following 2-step recurrence sequence

$$x(k+1) = \alpha * x(k) + \beta * x(k-1), \quad k \geq 2$$

   with the initial values $x(1)$ and $x(2)$ given.

   Test case : $x(1) = 1/3, x(2) = 1/12$, $\alpha = 2.25$ and $\beta = -1/2$. Compute the segment $x(30 : 60)$. The true sequence (and hence any of its segment) decreases with k in absolute value.

   If your program does not render the correct result, explain what goes wrong and try to fix the problem.