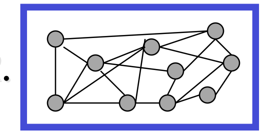
Today's topics

- Graphs
  - Basics & types
  - Properties
  - Connectivity
  - Hamilton & Euler Paths
- Reading: Sections 8.1-8.5

Simple Graphs

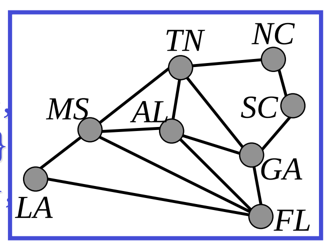
- Correspond to symmetric, irreflexive binary relations *R*.
- A *simple graph* *G*=(*V*,*E*) consists of:
  - a set *V* of *vertices* or *nodes* (*V* corresponds to the universe of the relation *R*),
  - a set *E* of *edges* / *arcs* / *links*: unordered pairs of [distinct] elements *u*,*v* ∈ *V*, such that *uRv*.



Visual Representation of a Simple Graph

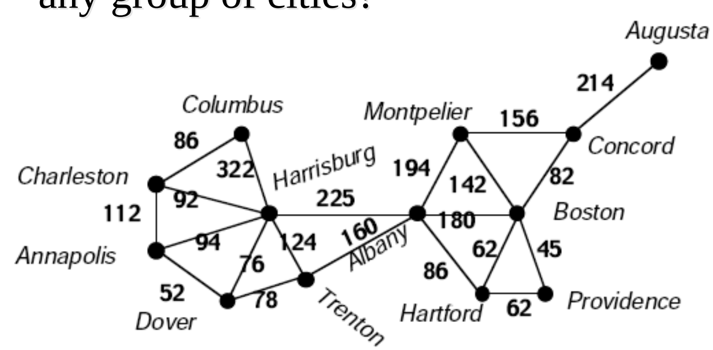
Example of a *Simple Graph*

- Let *V* be the set of states in the far-southeastern U.S.:
  - *I.e.*, *V*={FL, GA, AL, MS, LA, SC, TN, NC}
- Let *E*={{*u*,*v*}|*u* adjoins *v*}
  ={{FL,GA},{FL,AL},{FL,MS}, {FL,LA},{GA,AL},{AL,MS}, {MS,LA},{GA,SC},{GA,TN}, {SC,NC},{NC,TN},{MS,TN}, {MS,AL}}



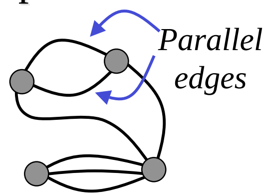
Graph example

- Can the edge weights below be correct for any group of cities?



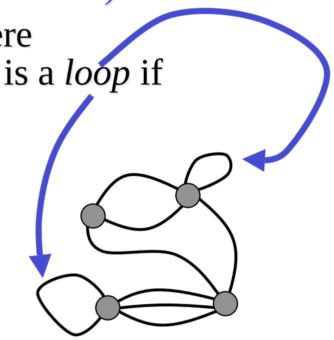
Multigraphs

- Like simple graphs, but there may be *more than one* edge connecting two given nodes.
- A *multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges (as primitive objects), and a function $f:E\rightarrow\{\{u,v\}|u,v\in V \wedge u\neq v\}$.
- *E.g.*, nodes are cities, edges are segments of major highways.



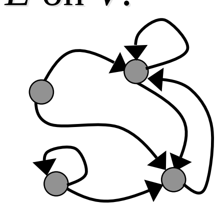
Pseudographs

- Like a multigraph, but edges connecting a node to itself are allowed. (R may be reflexive.)
- A *pseudograph* $G=(V, E, f)$ where $f:E\rightarrow\{\{u,v\}|u,v\in V\}$. Edge $e\in E$ is a *loop* if $f(e)=\{u,u\}=\{u\}$.
- *E.g.*, nodes are campsites in a state park, edges are hiking trails through the woods.



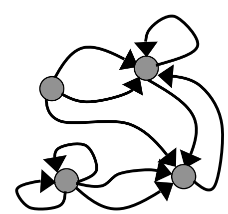
Directed Graphs

- Correspond to arbitrary binary relations R, which need not be symmetric.
- A *directed graph* (V,E) consists of a set of vertices V and a binary relation E on V.
- *E.g.*: V = set of People, $E=\{(x,y) \mid x \text{ loves } y\}$



Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- A *directed multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges, and a function $f:E\rightarrow V\times V$.
- *E.g.*, V=web pages, E=hyperlinks. *The WWW is a directed multigraph...*



Types of Graphs: Summary

- Summary of the book's definitions.
- Keep in mind this terminology is not fully standardized across different authors...

Term	Edge type	Multiple edges ok?	Self-loops ok?
Simple graph	Undir.	No	No
Multigraph	Undir.	Yes	No
Pseudograph	Undir.	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

§8.2: Graph Terminology

You need to learn the following terms:

- *Adjacent, connects, endpoints, degree, initial, terminal, in-degree, out-degree, complete, cycles, wheels, n-cubes, bipartite, subgraph, union.*

Adjacency

Let G be an undirected graph with edge set E. Let $e \in E$ be (or map to) the pair $\{u,v\}$. Then we say:

- u, v are *adjacent* / *neighbors* / *connected.*
- Edge e is *incident with* vertices u and v.
- Edge e *connects* u and v.
- Vertices u and v are *endpoints* of edge e.

Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The *degree* of v, $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
- A vertex with degree 0 is called *isolated*.
- A vertex of degree 1 is called *pendant*.

Handshaking Theorem

- Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- **Corollary:** Any undirected graph has an even number of vertices of odd degree.

Directed Adjacency

- Let G be a directed (possibly multi-) graph, and let e be an edge of G that is (or maps to) (u,v). Then we say:
  - u is *adjacent to* v, v is *adjacent from* u
  - *e comes from* u, e *goes to* v.
  - *e connects u to v*, *e goes from u to v*
  - the *initial vertex* of e is u
  - the *terminal vertex* of e is v

Directed Degree

- Let G be a directed graph, v a vertex of G.
  - The *in-degree* of v, $\deg^-(v)$, is the number of edges going to v.
  - The *out-degree* of v, $\deg^+(v)$, is the number of edges coming from v.
  - The *degree* of v, $\deg(v) :\equiv \deg^-(v) + \deg^+(v)$, is the sum of v's in-degree and out-degree.

Directed Handshaking Theorem

- Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.

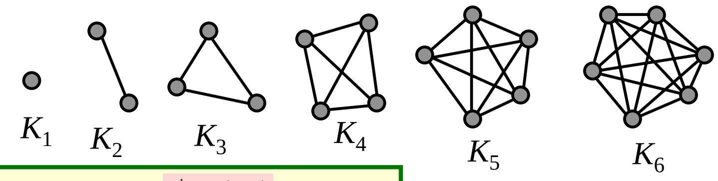
Special Graph Structures

Special cases of undirected graph structures:

- Complete graphs K_n
- Cycles C_n
- Wheels W_n
- n-Cubes Q_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$

Complete Graphs

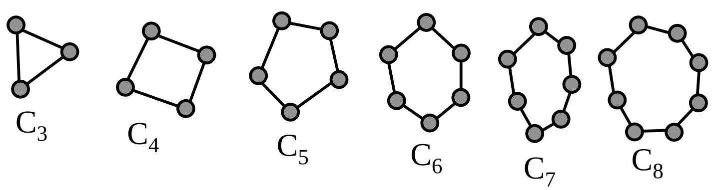
- For any $n \in \mathbf{N}$, a *complete graph* on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node: $\forall u,v \in V: u \neq v \leftrightarrow \{u,v\} \in E$.



Note that K_n has $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ edges.

Cycles

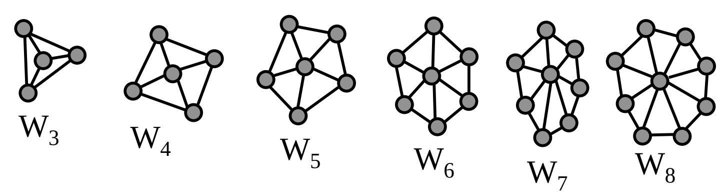
- For any $n \geq 3$, a *cycle* on n vertices, C_n, is a simple graph where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.



How many edges are there in C_n?

Wheels

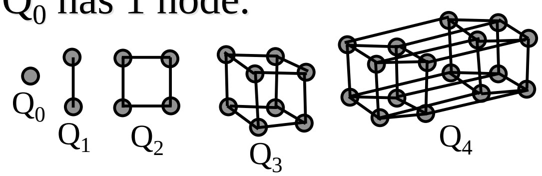
- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{hub},v_1\}, \{v_{hub},v_2\},\ldots,\{v_{hub},v_n\}\}$.



How many edges are there in W_n?

n-cubes (hypercubes)

- For any $n \in \mathbf{N}$, the hypercube Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



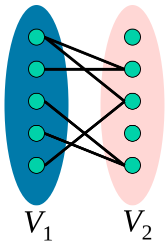
Number of vertices: 2^n. *Number of edges:Exercise to try!*

n-cubes (hypercubes)

- For any $n \in \mathbf{N}$, the hypercube Q_n can be defined recursively as follows:
  - $Q_0=\{\{v_0\},\emptyset\}$ (one node and no edges)
  - For any $n \in \mathbf{N}$, if $Q_n=(V,E)$, where $V=\{v_1,\ldots,v_a\}$ and $E=\{e_1,\ldots,e_b\}$, then $Q_{n+1}=(V\cup\{v_1',\ldots,v_a'\}, E \cup\{e_1',\ldots,e_b'\}\cup\{\{v_1,v_1'\},\{v_2,v_2'\},\ldots, \{v_a,v_a'\}\})$ where $v_1',\ldots,v_a'$ are new vertices, and where if $e_i=\{v_j,v_k\}$ then $e_i'=\{v_j',v_k'\}$.

Bipartite Graphs

- **Def'n.:** A graph $G=(V,E)$ is *bipartite* (two-part) iff $V = V_1 \cap V_2$ where $V_1 \cup V_2=\emptyset$ and $\forall e \in E: \exists v_1 \in V_1, v_2 \in V_2: e=\{v_1,v_2\}$.
- **In English:** The graph can be divided into two parts in such a way that all edges go between the two parts.

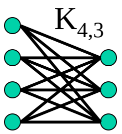


This definition can easily be adapted for the case of multigraphs and directed graphs as well.

Can represent with zero-one matrices.

Complete Bipartite Graphs

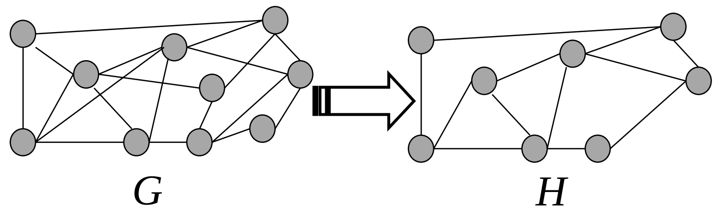
- For $m,n \in \mathbf{N}$, the *complete bipartite graph* $K_{m,n}$ is a bipartite graph where $|V_1| = m$, $|V_2| = n$, and $E = \{\{v_1,v_2\}|v_1 \in V_1 \wedge v_2 \in V_2\}$.
  - That is, there are *m* nodes in the left part, *n* nodes in the right part, and every node in the left part is connected to every node in the right part.



$K_{m,n}$ has _____ nodes and _____ edges.

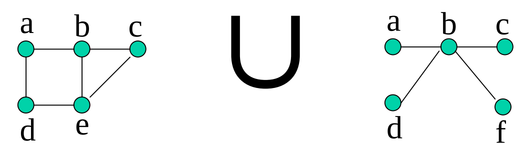
Subgraphs

- A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$.



Graph Unions

- The *union* $G_1 \cup G_2$ of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.

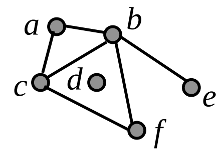


§8.3: Graph Representations & Isomorphism

- Graph representations:
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists

- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	*Adjacent Vertices*
a	*b, c*
b	*a, c, e, f*
c	*a, b, f*
d	
e	*b*
f	*c, b*

Directed Adjacency Lists

- 1 row per node, listing the terminal nodes of each edge incident from that node.

Adjacency Matrices

- A way to represent simple graphs
  - possibly with self-loops.
- Matrix $\mathbf{A}=[a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, and is 0 otherwise.
- Can extend to pseudographs by letting each matrix elements be the number of links (possibly >1) between the nodes.

Graph Isomorphism

- Formal definition:
  - Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f:V_1\rightarrow V_2$ such that $\forall\ a,b\in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
  - f is the "renaming" function between the two node sets that makes the two graphs identical.
  - This definition can easily be extended to other types of graphs.

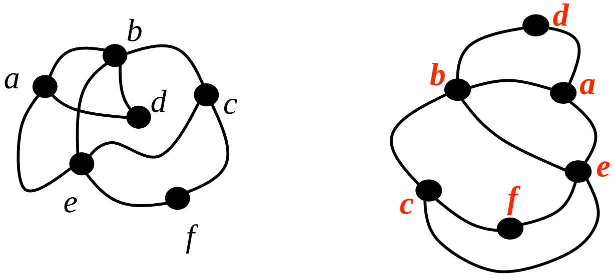
Graph Invariants under Isomorphism

Necessary but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

- We must have that $|V1|=|V2|$, and $|E1|=|E2|$.
- The number of vertices with degree n is the same in both graphs.
- For every proper subgraph g of one graph, there is a proper subgraph of the other graph that is isomorphic to g.

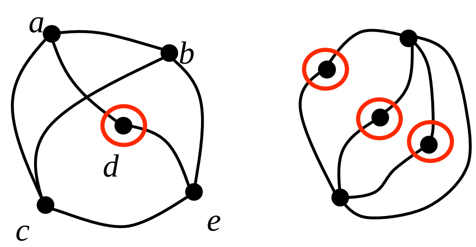
Isomorphism Example

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



- *Same # of vertices*
- *Same # of edges*
- *Different # of verts of degree 2! (1 vs 3)*

§8.4: Connectivity

- In an undirected graph, a *path of length n from u to v* is a sequence of adjacent edges going from vertex *u* to vertex *v*.
- A path is a *circuit* if *u*=*v*.
- A path *traverses* the vertices along it.
- A path is *simple* if it contains no edge more than once.

Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness

- An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- **Theorem:** There is a *simple* path between any pair of vertices in a connected undirected graph.
- *Connected component*: connected subgraph
- A *cut vertex* or *cut edge* separates 1 connected component into 2 if removed.

Directed Connectedness

- A directed graph is *strongly connected* iff there is a directed path from *a* to *b* for any two verts *a* and *b*.
- It is *weakly connected* iff the underlying *undirected* graph (*i.e.*, with edge directions removed) is connected.
- Note *strongly* implies *weakly* but not vice-versa.

Paths & Isomorphism

- Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.

Counting Paths w Adjacency Matrices

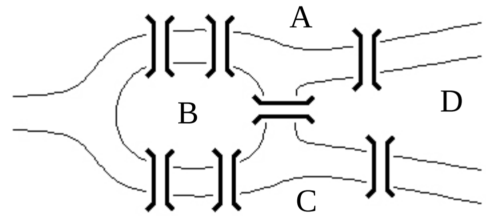
- Let **A** be the adjacency matrix of graph *G*.
- The number of paths of length *k* from v_i to v_j is equal to $(\mathbf{A}^k)_{i,j}$.
  - The notation $(\mathbf{M})_{i,j}$ denotes $m_{i,j}$ where $[m_{i,j}] = \mathbf{M}$.

§8.5: Euler & Hamilton Paths

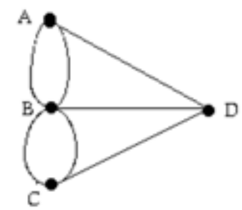
- An ***Euler circuit*** in a graph *G* is a simple circuit containing every edge of *G*.
- An ***Euler path*** in *G* is a simple path containing every edge of *G*.
- A ***Hamilton circuit*** is a circuit that traverses each vertex in *G* exactly once.
- A ***Hamilton path*** is a path that traverses each vertex in *G* exactly once.

Bridges of Königsberg Problem

- Can we walk through town, crossing each bridge exactly once, and return to start?



The original problem



Equivalent multigraph

Euler Path Theorems

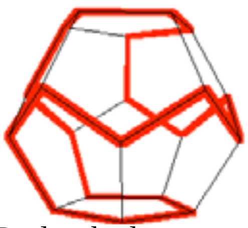
- **Theorem:** A connected multigraph has an Euler circuit iff each vertex has even degree.
  - **Proof:**
    - (→) The circuit contributes 2 to degree of each node.
    - (←) By construction using algorithm on p. 580-581
- **Theorem:** A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.
  - One is the start, the other is the end.

Euler Circuit Algorithm

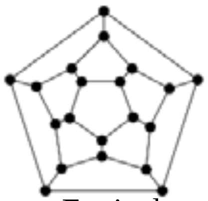
- Begin with any arbitrary node.
- Construct a simple path from it till you get back to start.
- Repeat for each remaining subgraph, splicing results back into original cycle.

Round-the-World Puzzle

- Can we traverse all the vertices of a dodecahedron, visiting each once?`



Dodecahedron puzzle



Equivalent graph

Pegboard version

Hamiltonian Path Theorems

- **Dirac's theorem**: If (but <u>not</u> only if) G is connected, simple, has $n \geq 3$ vertices, and $\forall v\ \deg(v) \geq n/2$, then G has a Hamilton circuit.
  - **Ore's corollary:** If G is connected, simple, has $n \geq 3$ nodes, and $\deg(u)+\deg(v) \geq n$ for every pair u,v of non-adjacent nodes, then G has a Hamilton circuit.