Today's topics

- Functions
  - Notations and terms
  - One-to-One vs. Onto
  - Floor, ceiling, and identity
- Reading: Sections 1.8
- Upcoming
  - Algorithms

On to section 1.8… Functions

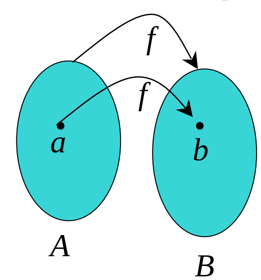
- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x\in\mathbf{R}$ a particular value $y=f(x)$, where $y\in\mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

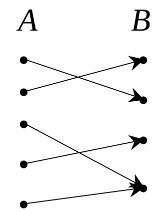
- For any sets A, B, we say that a *function f from (or "mapping") A to B* (f:$A\rightarrow B$) is a particular assignment of exactly one element $f(x)\in B$ to each element $x\in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function f assigns *zero or one* elements of B to each element $x\in A$.
  - Functions of n arguments; relations (ch. 6).

Graphical Representations

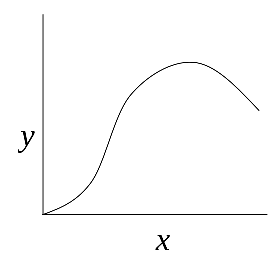
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from "situations" to truth values {**T**,**F**}
  - A logic system called *situation theory*.
  - p="It is raining."; s=our situation here,now
  - $p(s)\in\{\mathbf{T},\mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: *e.g.*, $\vee((\mathbf{F},\mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T},\mathbf{F})) = \mathbf{F}$.

A Neat Trick

- Sometimes we write Y^X to denote the set F of *all* possible functions f:$X\rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y, we have $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F}\equiv\mathbf{0}$, $\mathbf{T}\equiv\mathbf{1}$, $\mathbf{2}:\equiv\{\mathbf{0},\mathbf{1}\}=\{\mathbf{F},\mathbf{T}\}$, then a subset $T\subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If it is written that f:$A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then we say:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R\subseteq B$ of f is $R=\{b \mid \exists a\ f(a)=b\}$.

We also say the *signature* of f is $A\rightarrow B$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: ___________, and its range is _________.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is __________, but its codomain is ____________________.

Operators (general definition)

- An *n*-ary *operator over* (or *on*) the set *S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- *E.g.*, if $S=\{\mathbf{T},\mathbf{F}\}$, ¬ can be seen as a unary operator, and ∧,∨ are binary operators on *S*.
- Another example: ∪ and ∩ are binary operators on the set of all sets.

Constructing Function Operators

- If • ("dot") is any operator over *B*, then we can extend • to also denote an operator over functions $f:A\rightarrow B$.
- *E.g.*: Given any binary operator $\bullet: B\times B\rightarrow B$, and functions $f,g:A\rightarrow B$, we define $(f\bullet g):A\rightarrow B$ to be the function defined by: $\forall a\in A,\ (f\bullet g)(a) = f(a)\bullet g(a)$.

Function Operator Example

- +,× ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f,g:\mathbf{R}\rightarrow\mathbf{R}$:
  - $(f+g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f\times g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f\times g)(x) = f(x)\times g(x)$

Function Composition Operator

Note match here.

- For functions g:A→B and f:B→C, there is a special operator called *compose* ("○").
  - It <u>composes</u> (creates) a new function out of f and g by applying f to the result of applying g.
  - We say $(f○g)$:A→C, where $(f○g)(a) :\equiv f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that ○ (like Cartesian ×, but unlike +,∧,∪) is non-commuting. (Generally, $f○g \neq g○f$.)

Images of Sets under Functions

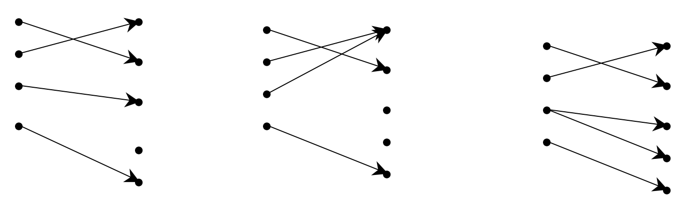
- Given f:A→B, and $S\subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s\in S\}$
  $:\equiv \{b \mid \exists\, s\in S: f(s)=b\}$.
- Note the range of f can be defined as simply the image (under f) of f's domain!

One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given f:A→B,
    "x is injective" :≡ $(\neg\exists x,y: x\neq y \wedge f(x)=f(y))$.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?
- Memory jogger: Each element of the domain is <u>injected</u> into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one — Not one-to-one — Not even a function!

Sufficient Conditions for 1-1ness

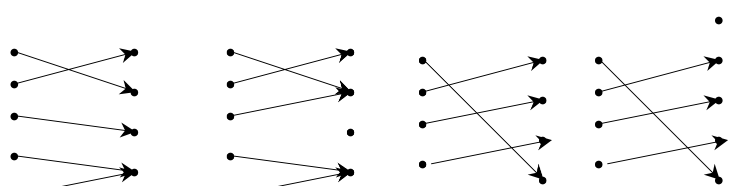
- For functions f over numbers, we say:
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true. E.g.* $1/x$

Onto (Surjective) Functions

- A function f:$A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A$: $f(a)=b$).
- Think: An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto

Bijections

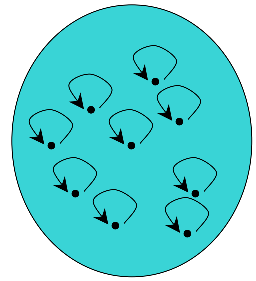
- A function f is said to be *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections f:$A \rightarrow B$, there exists an *inverse of* f, written f^{-1}:$B \rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$
  - (where I_A is the identity function on A)

The Identity Function

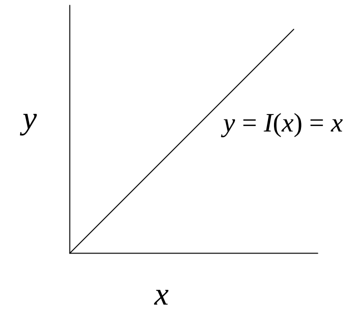
- For any domain A, the *identity function* $I:A \rightarrow A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A: I(a)=a$.
- Some identity functions you've seen:
  - +ing 0, ·ing by 1, ∧ing with **T**, ∨ing with **F**, ∪ing with ∅, ∩ing with *U*.
- Note that the identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range

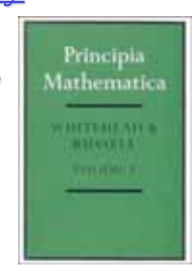


Graphs of Functions

- We can represent a function $f:A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$. ← The function's *graph*.
- Note that $\forall a$, there is only 1 pair (a,b).
  - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

Aside About Representations

- It is possible to represent any type of discrete structure (propositions, bit-strings, numbers, sets, ordered pairs, functions) in terms of virtually any of the other structures (or some combination thereof).
- Probably <u>none</u> of these structures is <u>truly</u> more fundamental than the others (whatever that would mean). However, strings, logic, and sets are often used as the foundation for all else. *E.g.* in ➔

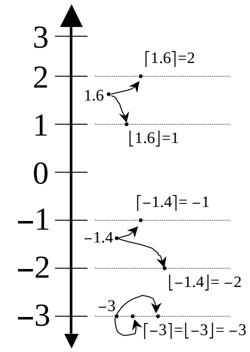


A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
  - The *floor* function $\lfloor\cdot\rfloor$:**R**→**Z**, where $\lfloor x\rfloor$ ("floor of x") means the largest (most positive) integer $\leq x$. *I.e.*, $\lfloor x\rfloor :\equiv \max(\{i\in\mathbf{Z} \mid i\leq x\})$.
  - The *ceiling* function $\lceil\cdot\rceil$:**R**→**Z**, where $\lceil x\rceil$ ("ceiling of x") means the smallest (most negative) integer $\geq x$. $\lfloor x\rfloor :\equiv \min(\{i\in\mathbf{Z} \mid i\geq x\})$

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x\notin\mathbf{Z}$, $\lfloor -x\rfloor \neq -\lfloor x\rfloor$ & $\lceil -x\rceil \neq -\lceil x\rceil$
- Note that if $x\in\mathbf{Z}$, $\lfloor x\rfloor = \lceil x\rceil = x$.



Plots with floor/ceiling

- Note that for $f(x)=\lfloor x\rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a\geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3\rfloor$: