Incentive Compatible Ranking Systems

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ABSTRACT

Ranking systems are a fundamental ingredient of basic e-commerce and Internet Technologies. In this paper we consider the issue of incentives in ranking systems, where agents act in order to maximize their position in the ranking, rather than to get a correct outcome. We consider two different notions of incentive compatibility and several basic properties of ranking systems, and show that in general no incentive compatible ranking system satisfying the conditions exist. However, we show that some artificial incentive compatible ranking systems do exist, satisfying only some of the properties.

Categories and Subject Descriptors

G.2.2 [**Discrete Mathematics**]: Graph Theory; J.4 [**Social and Behavioral Sciences**]: Economics
; H.3.3 [**Information Storage and Retrieval**]: Information Search and Retrieval

General Terms

Algorithms, Economics, Human Factors, Theory

Keywords

Ranking systems, multi-agent systems, incentives, social choice

1. INTRODUCTION

The ranking of agents based on other agents' input is fundamental to e-commerce and multi-agent systems (see e.g. [5, 14]). Moreover, the ranking of agents based on other agents' input have become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google's PageRank algorithm[11] and ebay's reputation system[13].

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This basic problem introduces a new social choice model. In the classical theory of social choice, as manifested by Arrow[3], a set of agents/voters is called to rank a set of alternatives. Given the agents' input, i.e. the agents' individual rankings, a social ranking of the alternatives is generated. The theory studies desired properties of the aggregation of agents' rankings into a social ranking. In particular, Arrow's celebrated impossibility theorem[3] shows that there is no aggregation rule that satisfies some minimal requirements, while by relaxing any of these requirements appropriate social aggregation rules can be defined. The novel feature of the ranking systems setting is that the set of agents and the set of alternatives **coincide**. Therefore, in such setting agents consider the effect of their votes on their own ranking, sometimes even more than they consider the effect the vote has on the agents actually voted for.

Notice that a natural interpretation/application of this setting is the ranking of Internet pages. In this case, the set of agents represents the set of Internet pages, and the links from a page p to a set of pages Q can be viewed as a two-level ranking where agents in Q are preferred by agent(page) p to the agents(pages) which are not in Q. The problem of finding an appropriate social ranking in this case is in fact the problem of (global) page ranking. Particular approaches for obtaining a useful page ranking have been implemented by search engines such as Google[11].

In the ranking systems setting, an agent(page) can only vote(link) to a set of agents(pages), leading to a dichotomous setting in which classical impossibility results do not apply. In particular, in the classical dichotomous social choice setting, the *Approval Voting* social choice rule satisfies all classical requirements of Arrow's and similar impossibility theorems.

The theory of social choice consists of two complementary axiomatic perspectives:

- The descriptive perspective: given a particular rule r for the aggregation of individual rankings into a social ranking, find a set of axioms that are sound and complete for r. That is, find a set of requirements that r satisfies; moreover, every social aggregation rule that satisfies these requirements should coincide with r. A result showing such an axiomatization is termed a *representation theorem* and it captures the exact essence of (and assumptions behind) the use of the particular rule.

- The normative perspective: devise a set of requirements that a social aggregation rule should satisfy, and try to find whether there is a social aggregation rule that satisfies these requirements.

Many efforts have been invested in the descriptive approach in the framework of the classical theory of social choice. In that setting, representation theorems have been presented for major voting rules such as the majority rule[9] (see [10] for an overview). Recently, we have successfully applied the descriptive perspective in the context of ranking systems by providing a representation theorem[2] for the well-known PageRank algorithm [11], which is the basis of Google's search technology. Still in the descriptive approach, various known ranking systems have been recently compared with regard to certain criteria by [4], and several ranking rules have been axiomatized [12, 16].

An excellent example for the normative perspective is Arrow's impossibility theorem[3]. In [1], we have proven an impossibility theorem for ranking systems where the set of voters and the set of alternatives coincide, when assuming two moderately strong axioms.

Although the above mentioned work consists of a significant body of rigorous research on ranking systems, the study did not consider the effects of the agents' incentives on ranking systems. The issue of incentives has been extensively studied in the classical social choice literature. The Gibbard–Satterthwaite theorem[7, 15] shows that in the classical social welfare setting, it is impossible to aggregate the rankings in a strategyproof fashion under some basic conditions. The incentives of the candidates themselves were also considered in the context of elections[6], where a related impossibility result is presented. However, these impossibility results do not apply to the ranking systems setting due to its dichotomous nature.

In this paper we consider the issue of incentives in ranking systems. We define a notion of strong incentive compatibility, where an agent is concerned with its exact position in the ranking, and a notion of k-incentive compatibility, where the agent is concerned in its *expected* position in the ranking with tolerance of k. We see that when we assume some very basic properties, such 0-incentive compatible ranking systems do not exist. However, if we assume only some of these properties, some artificial incentive compatible ranking systems do exist.

Our results expose some surprising and illuminating effects of some basic properties one may require a ranking system to satisfy on the existence of incentive compatible ranking systems.

This paper is structured as follows: In Section 2 we formally introduce the notion of ranking systems and in Section 3 we define some basic properties of ranking systems. In Section 4 we introduce our two notions of incentive compatibility. We then show a strong possibility result in Section 5, when we do not assume the *minimal fairness* property. In Section 6 we provide a full classification of the existence of incentive compatible ranking systems when we do assume minimal fairness. Section 7 provides some illuminating lessons learned from this classification. Finally, in Section 8 we introduce the isomorphism property and recommend further research with regard to the classification of incentive compatibility under isomorphism.

2. RANKING SYSTEMS

Before describing our results regarding ranking systems, we must first formally define what we mean by the words "ranking system" in terms of graphs and linear orderings:

DEFINITION 1. *Let A be some set. A relation $R \subseteq A \times A$ is called an* ordering *on A if it is reflexive, transitive, and complete. Let $L(A)$ denote the set of orderings on A.*

NOTATION 1. *Let $\preceq$ be an ordering, then $\simeq$ is the equality predicate of $\preceq$, and $\prec$ is the strict order induced by $\preceq$. Formally, $a \simeq b$ if and only if $a \preceq b$ and $b \preceq a$; and $a \prec b$ if and only if $a \preceq b$ but not $b \preceq a$.*

Given the above we can define what a ranking system is:

DEFINITION 2. *Let $\mathbb{G}_V$ be the set of all graphs on a vertex set V that do not include self edges*[1]*.* A ranking system *F is a functional that for every finite vertex set V maps graphs $G \in \mathbb{G}_V$ to an ordering $\preceq^F_G \in L(V)$.*

One can view this setting as a variation/extension of the classical theory of social choice as modeled by [3]. The ranking systems setting differs in two main properties. First, in this setting we assume that the set of voters and the set of alternatives coincide, and second, we allow agents only two levels of preference over the alternatives, as opposed to Arrow's setting where agents could rank alternatives arbitrarily.

3. BASIC PROPERTIES OF RANKING SYSTEMS

Now we define some basic properties of ranking systems to guide our classification. Most properties have two versions – one weak and one strong.

First of all, we define the notion of a trivial ranking system, which ranks any two vertices the same way in all graphs.

DEFINITION 3. *A ranking system F is called* trivial *if for all vertices v_1, v_2 and for all graphs G, G' which include these vertices: $v_1 \preceq^F_G v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$. A ranking system F is called* nontrivial *if it is not trivial.*

A ranking system F is called infinitely nontrivial *if there exist vertices v_1, v_2 such that for all $N \in \mathbb{N}$ there exists $n > N$ and graphs $G = (V, E)$ and $G' = (V', E')$ s.t. $|V| = |V'| = n$, $v_1 \preceq^F_G v_2$, but $v_2 \prec^F_{G'} v_1$.*

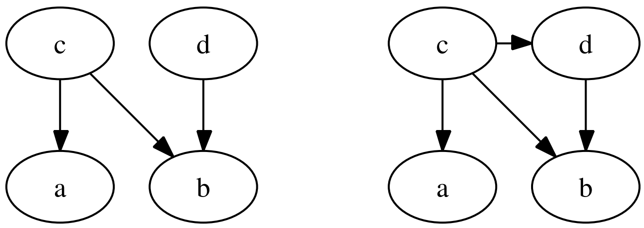
A basic requirement from a ranking system is that when there are no votes in the system, all agents must be ranked equally. We call this requirement minimal fairness[2].

DEFINITION 4. *A ranking system F is* minimally fair *if for every graph $G = (V, \emptyset)$ with no edges, and for every $v_1, v_2 \in V$: $v_1 \simeq^F_G v_2$.*

Another basic requirement from a ranking system is that as agents gain additional votes, their rank must improve, or at least not worsen. Surprisingly, this vague notion can be formalized in (at least) two distinct ways: The monotonicity property considers the situation where one agent has a

[1]Our results are still correct when allowing self-edges, but for the simplicity of the exposition we assume none exist.

[2]A stronger notion of fairness, the isomorphism property, will be considered in Section 8.



(a) G_1 (b) G_2

Figure 1: Example graphs for the basic properties of ranking systems

superset of the votes another has *in the same graph*, where the positive response property considers the addition of a vote for an agent *between graphs*. This distinction is important because, as we will see, the two properties are neither equivalent, nor imply each other.

NOTATION 2. *Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex. The predecessor set of v is $P_G(v) = \{v'|(v', v) \in E\}$. The successor set of v is $S_G(v) = \{v'|(v, v') \in E\}$.*

DEFINITION 5. *Let F be a ranking system. F satisfies* weak positive response *if for all graphs $G = (V, E)$ and for all $(v_1, v_2) \in (V \times V) \setminus E$, and for all $v_3 \in V$: Let $G' = (V, E \cup (v_1, v_2))$. Then, $v_3 \preceq^F_G v_2$ implies $v_3 \leq^F_{G'} v_2$ and $v_3 <^F_G v_2$ implies $v_3 <^F_{G'} v_2$. F furthermore satisfies* strong positive response *if $v_3 \preceq^F_G v_2$ implies $v_3 <^F_{G'} v_2$.*

DEFINITION 6. *A ranking system F satisfies* weak monotonicity *if for all $G = (V, E)$ and for all $v_1, v_2 \in V$: If $P(v_1) \subseteq P(v_2)$ then $v_1 \preceq^F_G v_2$. F furthermore satisfies* strong monotonicity *if $P(v_1) \subsetneq P(v_2)$ additionally implies $v_1 \prec^F_G v_2$.*

EXAMPLE 1. *Consider the graphs G_1 and G_2 in Figure 1. Further assume a ranking system F ranks $a \simeq^F_{G_1} d$ in graph G_1. Then, if F satisfies weak positive response, it must also rank $a \preceq^F_{G_2} d$ in G_2. If F satisfies the strong positive response, then it must strictly rank $a \prec^F_{G_2} d$ in G_2. However, if we do not assume $a \preceq^F_{G_1} d$, F may rank a and d arbitrarily in G_2.*

Now consider the graph G_1, and note that $P(a) = \{c\} \subsetneq \{c, d\} = P(b)$. This is the requirement of the weak (and strong) monotonicity property, and thus any ranking system F that satisfies weak monotonicity must rank $a \preceq^F_{G_1} b$, and it is satisfies strong monotonicity, it must strictly rank $a \prec^F_{G_1} b$.

Note that the weak monotonicity property implies minimal fairness. This is due to the fact that when no votes are cast, all vertices have exactly the same predecessor sets and thus must be ranked equally.

Yet another simple requirement from a ranking system is that it does not behave arbitrarily differently when two sets of agents with their respective votes are considered one set.

DEFINITION 7. *Let F be ranking system and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs s.t. $V_1 \cap V_2 = \emptyset$ and let $v_1, v_2 \in V_1$ be two vertices. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. F satisfies the* weak union condition *if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G_1 \cup G_2} v_2$. Let $G' = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$, where $E \subseteq V_1 \times V_2$ is in an arbitrary set of edges from V_1 to V_2. F satisfies the* strong union condition *if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$.*

Surprisingly, we will see that even the weak union condition has great significance towards the existence of a ranking system or lack thereof. One reason for this effect, is that a ranking system satisfying this condition cannot behave differently depending on the size of the graph.

3.1 Satisfiability

Now that we have defined some properties, the question arises whether these properties can be satisfied simultaneously, and if so, which known ranking systems have which properties.

It turns out that, with the exception of the strong union condition, all the properties above are satisfied by almost all known ranking systems such as the PageRank[11] ranking system (with a damping factor) and the authority ranking by the Hubs&Authorities algorithm[8]. These ranking systems *do not* satisfy the strong union condition, as in both systems outgoing links outside an agent's strongly connected component may affect ranks inside the strongly connected component, either by dividing the importance (in PageRank) or by affecting the hubbiness score in Hubs&Authorities.

Furthermore, the simple *approval voting* ranking system satisfies all the strong properties mentioned above. The approval voting ranking system can be defined as follows:

DEFINITION 8. *The* approval voting ranking system *AV is the ranking system defined by:*

$$v_1 \preceq^{AV}_G v_2 \Leftrightarrow |P(v_1)| \leq |P(v_2)|.$$

FACT 1. *The approval voting ranking system AV satisfies minimal fairness, strong monotonicity, strong positive response, the strong union condition, and infinite nontriviality.*

The proof is this fact is left as an exercise to the reader.

These facts lead us to believe that the properties defined above (perhaps with the exception of the strong union condition), should all be satisfied by any reasonable ranking system, at least in their weak form. We will soon show that this is not possible when requiring incentive compatibility.

4. INCENTIVE COMPATIBILITY

Ranking systems do not exist in empty space. The results given by ranking systems frequently have implications for the agents being ranked, which are the same agents that are involved in the ranking. Therefore, the incentives of these agents should in many cases be taken into consideration.

In our approach, we require that our ranking system will not rank agents better for stating untrue preferences, but we assume that the agents are interested only in their own ranking (and not, say, in the ranking of those they prefer).

The strong incentive compatibility property further assumes a strong preference of the agents with regard to their rank: Each agent would like to minimize the number of agents ranked higher than herself, and then minimize the number of agents ranked the same as herself.

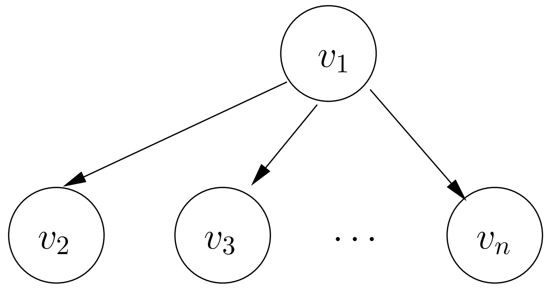


Figure 2: Proof that AV is not $(\frac{n}{2}-1)$-incentive compatible

Definition 9. *Let F be a ranking system. F satisfies* strong incentive compatibility *if for all true preference graphs $G = (V, E)$, for all vertices $v \in V$, and for all preferences $V_v \subseteq V$ reported by v: Let $E' = E \setminus \{(v, x)|x \in V\} \cup \{(v, x)|x \in V_v\}$ and $G' = (V, E')$ be the reported preference graph. Then, $|\{x \in V|v \prec^F_{G'} x\}| \geq |\{x \in V|v \prec^F_G x\}|$; and if $|\{x \in V|v \prec^F_{G'} x\}| = |\{x \in V|v \prec^F_G x\}|$ then $|\{x \in V|v \simeq^F_{G'} x\}| \geq |\{x \in V|v \simeq^F_G x\}|$.*

We can weaken this requirement by assuming that an agent in interested in its *expected* rank, assuming equally ranked agents are ordered randomly. Formally, the expected rank (henceforth referred to simply as *rank*) is defined as follows:

Definition 10. *The* rank *of a vertex v in a graph G under the ranking system F is defined as*

$$r^F_G(v) = \tfrac{1}{2}\left|\{v'|v' \prec v\}\right| + \tfrac{1}{2}\left|\{v'|v' \preceq v\}\right|.$$

Given this definition of rank, we can now require that an agent cannot improve its rank by more than some constant k.

Definition 11. *Let F be ranking system and let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be graphs s.t. for some $v \in V$, and for all $v' \in V \setminus \{v\}, v'' \in V : (v', v'') \in E_1 \Leftrightarrow (v', v'') \in E_2$. F is called k-*incentive compatible *if $\left|r^F_{G_2}(v) - r^F_{G_1}(v)\right| \leq k$.*

In the rest of this paper we will focus on 0-incentive compatible ranking systems and those that satisfy strong incentive compatibility. Interestingly, we will see in the remainder of this paper that these incentive compatibility properties are very hard to satisfy, and no common nontrivial ranking system satisfies them. In particular, the PageRank, Hubs&Authorities and Approval Voting ranking systems mentioned above are not 0-incentive compatible. In fact, these systems are not even $(\frac{n}{2}-1)$-incentive compatible when n is the number of vertices in the graph.

Example 2. *To see that the approval voting ranking system AV is not $(\frac{n}{2}-1)$-incentive compatible, consider a graph G with n vertices $\{v_1, \ldots, v_n\}$, where v_1 points to $\{v_2, \ldots, v_n\}$, and no other edges exist, as illustrated in Figure 2. In this graph, AV ranks $v_1 \preceq v_2 \simeq \cdots \simeq v_n$, and thus $r^{AV}_G(v_1) = \frac{1}{2}$. However in the graph G' where v_1 does not point to any vertex, the ranking is be $v_1 \simeq v_2 \simeq \cdots \simeq v_n$, and thus $r^{AV}_{G'}(v_1) = \frac{n}{2}$. We see that v_1 can improve its rank by $\frac{n-1}{2}$, and thus the ranking system cannot be $(\frac{n}{2}-1)$-incentive compatible.*

5. POSSIBILITY WITHOUT MINIMAL FAIRNESS

To begin our classification of the existence of incentive compatible ranking systems, we first consider ranking systems which do not satisfy minimal fairness. We have already seen that minimal fairness is implied by weak monotonicity, so we cannot hope to be satisfy weak monotonicity without minimal fairness. As it turns out, the strong versions of all the remaining properties considered above can, in fact, be satisfied simultaneously.

Proposition 1. *There exists a ranking system F_1 that satisfies strong incentive compatibility, strong positive response, infinite nontriviality, and the strong union condition.*

Proof. *Assume a lexicographic order $<$ over vertex names, and assume three consecutive vertices $v_1 < v_2 < v_3$. Then, F_1 is defined as follows (let $G = (V, E)$ be some graph):*

$$\begin{aligned} v \preceq^{F_1}_G u \quad \Leftrightarrow \quad & [v \leq u \wedge (v \neq v_2 \vee u \neq v_3)] \vee \\ & [v = v_2 \wedge u = v_3 \wedge (v_1, v_2) \notin E] \vee \\ & [v = v_3 \wedge u = v_2 \wedge (v_1, v_2) \in E]. \end{aligned}$$

That is, vertices are ranked strictly according to their lexicographic order, except when $(v_1, v_2) \in E$, whereas the ranking of v_2 and v_3 is reversed.

F_1 is infinitely nontrivial because graphs with the vertices v_1, v_2, v_3 are ranked differently depending on the existence of the edge (v_1, v_2), and these exist for any $|V| \geq 3$.

F_1 satisfies strong incentive compatibility because the only vertex that can make any change in the ranking is v_1 and it cannot ever change its own position in the ranking at all.

F_1 satisfies strong positive response because the ordering of the vertices remains unchanged by anything but the (v_1, v_2) edge, and is always strict. The addition of the (v_1, v_2) edge only increases the relative rank of v_2 as required.

Assume for contradiction that F_1 does not satisfy the strong union condition. Then, there exist two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and an edge set $E \subseteq V_1 \times V_2$ such that the ranking $\preceq^{F_1}_G$ of graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$ is inconsistent with $\preceq^{F_1}_{G_1}$. First note that the only inconsistency that may arise is with the ranking of v_2 compared to v_3. Therefore, $\{v_2, v_3\} \subseteq V_1$. Furthermore, for the ranking to be inconsistent $(v_1, v_2) \notin E_1$ and $(v_1, v_2) \in E_1 \cup E_2 \cup E$ (the opposite is impossible due to inclusion). Furthermore, $v_2 \in V_1 \Rightarrow v_2 \notin V_2 \Rightarrow (v_1, v_2) \notin V_1 \times V_2 \Rightarrow (v_1, v_2) \notin E$. Thus we conclude that $(v_1, v_2) \in E_2$, and thus $v_2 \in V_2$, in contradiction to the fact that $v_2 \in V_1$. □

6. FULL CLASSIFICATION UNDER MINIMAL FAIRNESS

We are now ready to state our main results:

Theorem 1. *There exist 0-incentive compatible, infinitely nontrivial, minimally fair ranking systems F_2, F_3, F_4 that satisfy either one of the three properties: weak monotonicity; weak positive response; and the weak union condition. However, there is no 0-incentive compatible, nontrivial, minimally fair ranking system that satisfies any two of those properties.*

THEOREM 2. *There is no 0-incentive compatible, nontrivial, minimally fair ranking system that satisfies either one the four properties: strong monotonicity, strong positive response, the strong union condition and strong incentive compatibility.*

The proof of these two theorems is split into ten different cases that must be considered – three possibility proofs for F_2, F_3. and F_4, three impossibility results with pairs of weak properties, and four impossibility results with each of the strong properties. We will now prove each of these cases.

6.1 Possibility Proofs

PROPOSITION 2. *There exists a 0-incentive compatible ranking system F_2 that satisfies minimal fairness, weak positive response, and infinite nontriviality.*

PROOF. *Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_2 is defined as follows:*

$$v \preceq u \quad \Leftrightarrow \quad [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee (v_1, v_3) \notin E \vee v_2 \notin V.$$

That is, F_2 ranks all vertices equally, except when the edge (v_1, v_3) exists. Then, F_2 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_2 satisfies minimal fairness because when no edges exist, the clause $(v_1, v_3) \notin E$ always matches, and thus all vertices are ranked equally, as required. F_2 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edge (v_1, v_3), which is ranked nontrivially.

F_2 satisfies weak positive response because the only edge addition that changes the ranks of the vertices in the graph (the addition of (v_1, v_3)) indeed doesn't weaken the target vertex v_3.

F_2 is 0-incentive compatible because only v_1 can affect the ranking of the vertices in the graph (by voting for v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. $\square$

PROPOSITION 3. *There exists a 0-incentive compatible ranking system F_3 that satisfies minimal fairness, the weak union condition, and infinite nontriviality.*

PROOF. *Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_3 is defined as follows:*

$$v \preceq u \quad \Leftrightarrow \quad [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee \{(v_1, v_2), (v_1, v_3)\} \not\subseteq E.$$

That is, F_3 ranks all vertices equally, except when the edges $(v_1, v_2), (v_1, v_3)$ exist. Then, F_3 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_3 satisfies minimal fairness because when no edges exist, the clause $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E$ always matches, as required. F_3 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edges $\{(v_1, v_2), (v_1, v_3)\}$, which is ranked nontrivially.

To prove F_3 satisfies the weak union condition, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be some graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. If $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E_1 \cup E_2$ then by the definition of F_3, it must rank all vertices in all graphs G_1, G_2, G equally, as required. Otherwise, for all $v, u \in (V_1 \cup V_2) \setminus \{v_2, v_3\}$: $v_2 \prec_G^{F_3} v \simeq_G^{F_3} u \prec_G^{F_3} v_3$. Assume wlog that $(v_1, v_2) \in E_1$ and thus $v_1, v_2 \in V_1$. But then also $(v_1, v_3) \in E_1$ and thus also $v_3 \in V_1$. By the definition of F_3, for all $v, u \in V_1 \setminus \{v_2, v_3\}$: $v_2 \prec_{G_1}^{F_3} v \simeq_{G_1}^{F_3} u \prec_{G_1}^{F_3} v_3$. As $v_1, v_2, v_3 \notin G_2$, trivially for all $v, u \in V_2$: $v \simeq_{G_2}^{F_3} u$, as required.

F_3 is 0-incentive compatible because only v_1 (if at all) can affect the ranking of the vertices in the graph (by voting for v_2 and v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. $\square$

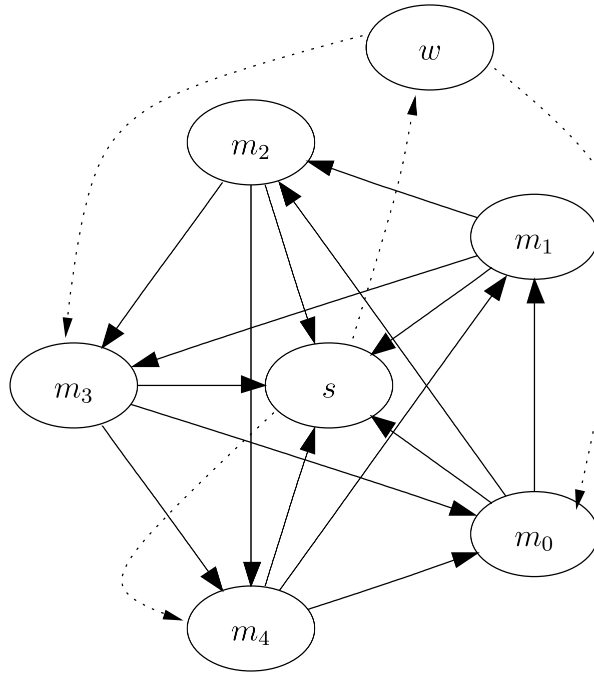


Figure 3: Nontrivially ranked graph for F_4

PROPOSITION 4. *There exists a 0-incentive compatible ranking system F_4 that satisfies minimal fairness, weak monotonicity, and infinite nontriviality.*

PROOF. *The ranking system F_4 ranks all vertices equally, except for graphs $G = (V, E)$ for which $|V| \geq 7$, $V = \{w, s, m_0, \ldots, m_{n-1}\}$, and for all $i \in \{0, \ldots, n-1\}$: $(m_i, s) \in E$, $(m_i, w) \notin E$, and for all $j \in \{0, \ldots, n-1\}$: $(m_i, m_j) \in E$ if and only if $j = (i+1) \bmod n$ or $j = (i+2) \bmod n$. Figure 3 includes an example graph that satisfies these conditions. In such graphs, F_4 ranks $w \prec_G^{F_4} m_1 \simeq_G^{F_4} \cdots \simeq_G^{F_4} m_n \prec_G^{F_4} s$.*

F_4 is minimally fair because when there are no edges, all vertices are ranked equally. F_4 satisfies infinite nontriviality because such nontrivially ranked graphs G exist for all $|V| \geq 7$.

F_4 satisfies weak monotonicity because in the graphs that it doesn't rank all vertices equally we see that $P(w) \not\supseteq P(m_i) \not\supseteq P(s)$ for all $i \in \{0, \ldots n-1\}$, which is consistent with the ordering F_4 specifies.

To prove F_4 is 0-incentive compatible, we let G_1, G_2 be two graphs that differ only in the outgoing edges of a single vertex v, and show that $r_{G_1}^{F_4}(v) = r_{G_2}^{F_4}(v)$. Because all graphs in which not all vertices are ranked equally are of the form defined above, at least one of the graphs G_1, G_2 must have this form. Let us assume wlog that this graph is G_1, and mark the vertices of this graph as defined above.

Now consider two cases:

1. *If $v = w$ or $v = s$, then by the definition of F_4, $\preceq^{F_4}_{G_1} \equiv \preceq^{F_4}_{G_2}$, thus trivially, $r^{F_4}_{G_1}(v) = r^{F_4}_{G_2}(v)$, as required.*
2. *If $v = m_i$ for some $i \in \{0, \ldots, n-1\}$, then first note that $r^{F_4}_{G_1}(v) = \frac{|V|}{2}$. If G_2 is not of the form defined above then all its vertices are ranked equally and specifically $r^{F_4}_{G_2} = \frac{|V|}{2}$, as required. Otherwise, G_2 is of the form defined above. Let w' and s' be the w and s vertices for G_2 in the form defined above. By the definition, $2 \leq |P_{G_1}(v)| \leq 4$, while $|P_{G_2}(w')| \leq 1$ and $|P_{G_2}(s')| \geq 5$. Therefore, $v \notin \{w', s'\}$. By the definition of F_4, $r^{F_4}_{G_2}(v) = \frac{|V|}{2}$, as required.*

□

6.2 Impossibility proofs with pairs of weak properties

We prove the impossibility results with pairs of weak properties, by assuming existence of a ranking system and analyzing the minimal graph in which the ranking system does not rank all agents equally. This is done in the following lemma:

LEMMA 1. *Let F be a 0-incentive compatible minimally fair nontrivial ranking system. Then, there exists a graph $G = (V, E)$ and vertices $v_\perp, v_\top, v \in V$ such that:*

1. *For all graphs $G' = (V', E')$ where $|E'| < |E|$ or $|E'| = |E|$ and $|V'| < |V|$, $v_1 \simeq^F_{G'} v_2$ for all $v_1, v_2 \in V'$.*
2. *$r^F_G(v) = \frac{|V|}{2}$*
3. *$v_\perp \prec^F_G v \prec^F_G v_\top$*
4. *For all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$.*
5. *$S(v) \neq \emptyset$ and for all $v' \in V$ such that $S(v') \neq \emptyset$: $v' \simeq^F_G v$.*

PROOF. *Let $G = (V, E)$ be a minimal (in edges, then vertices) graph such that there exist v_1, v_2 where $v_1 \prec^F_G v_2$. Such a graph exists because F is nontrivial. This graph immediately satisfies condition 1. Let $v_\perp, v_\top$ be vertices such that for all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$ (such vertices exist because $\preceq$ is an ordering). Note that these vertices satisfy condition 4.*

$E \neq \emptyset$ because minimal fairness will force $v_1 \simeq v_2$. Let $(v, v') \in E$ be some edge. From minimallity, $r^F_{(V, E \setminus \{(v,v')\})}(v) = \frac{|V|}{2}$. From 0-incentive compatibility, $r^F_G(v) = \frac{|V|}{2}$, satisfying condition 2. Therefore,

$$\begin{aligned}
\tfrac{1}{2}\left|\{v'|v' \prec v\}\right| + \tfrac{1}{2}\left|\{v'|v' \preceq v\}\right| &= \tfrac{1}{2}|V| \\
\left|\{v'|v' \prec v\}\right| + \left|\{v'|v' \preceq v\}\right| &= \left|\{v'|v' \preceq v\}\right| + \\
&\quad + \left|\{v'|v' \succ v\}\right| \\
\left|\{v'|v' \prec v\}\right| &= \left|\{v'|v' \succ v\}\right|.
\end{aligned}$$

From the assumption that $v_1 \prec^F_G v_2$: $v_\perp \preceq^F_G v_1 \prec^F_G v_2 \preceq^F_G v_\top$. Therefore, $v_\perp \prec v$ or $v \prec v_\top$. But as $|\{v'|v' \prec v\}| = |\{v'|v' \succ v\}|$, and at least one is nonempty, both $v_\perp \prec v \prec v_\top$, satisfying condition 3.

Condition 5 is satisfied by noting that for all v' such that $S(v) \neq \emptyset$, $r^F_G(v') = \frac{|V|}{2} = r^F_G(v)$, and thus $v' \simeq^F_G v$. □

Now we can prove the impossibility results for any pair of weak properties:

PROPOSITION 5. *There exists no 0-incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak positive response conditions.*

PROOF. *Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma.*

Now, let $(v_1, v_2) \in E$ be some edge. Let $G' = (V, E \setminus \{(v_1, v_2)\})$. By condition 1, $v_2 \simeq^F_{G'} v_\top$. By weak positive response, $v_\top \preceq^F_G v_2$. Since this is true for all $v_2 \in V$ with $P(v_2) = \emptyset$, and $v_\perp \prec^F_G v \prec^F_G v_\top$, we conclude that $P_G(v_\perp) = P_G(v) = \emptyset$. Now, by weak monotonicity $v_\perp \simeq^F_G v$, in contradiction to the fact that $v_\perp \prec^F_G v$. □

PROPOSITION 6. *There exists no 0-incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak union conditions.*

PROOF. *Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma.*

Now let $G' = (V \cup \{x\}, E)$ be a graph with an additional vertex $x \notin V$. By the weak union condition, $v_\perp \prec^F_{G'} v$. By weak monotonicity, $x \preceq^F_{G'} v_\perp$. Therefore, by the weak union condition, $r^F_{G'}(v) = r^F_G(v) + 1 = \frac{|V|}{2} + 1$. Let $G'' = (V \cup \{x\}, E \setminus \{(v', v)|v' \in V\})$. By condition 1 and the fact that $S_{G'}(v) \neq \emptyset$, $r^F_{G''}(v) = \frac{|V|+1}{2}$. From 0-incentive compatibility, $r^F_{G''}(v) = r^F_{G'}(v)$, which is a contradiction. □

PROPOSITION 7. *There exists no 0-incentive compatible nontrivial minimally fair ranking system that satisfies the weak union and weak positive response conditions.*

PROOF. *Assume for contradiction a ranking system F that satisfies the conditions. As the conditions of Lemma 1 are satisfied, let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma. Now let $G_1 = (V \setminus \{v_\perp\}, E)$ and let $G_2 = (\{v_\perp\}, \emptyset)$. From conditions 3 and 5, $S(v_\perp) = \emptyset$. If $P_G(v_\perp) \neq \emptyset$, then by condition 1 in the graph $G' = (V, E \setminus \{(x, v_\perp)\})$ where $x \in P_G(v_\perp)$, $v_\top \preceq^F_{G'} v_\perp$. But then by weak positive response $v_\top \preceq^F_G v_\perp$ in contradiction to condition 3.*

Therefore, $P_G(v_\perp) = S_G(v_\perp) = \emptyset$. Thus, G_1 and G_2 satisfy the conditions of the weak union condition with regard to G. Therefore, $v \prec^F_G v_\top \Rightarrow v \prec^F_{G_1} v_\top$, in contradiction to condition 1, because the edge set is the same and $|V_1| < |V|$. □

6.3 Impossibility proofs with the strong properties

PROPOSITION 8. *There exists no 0-incentive compatible minimally fair ranking system that satisfies strong positive response.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. Assume a graph G with two vertices $V = \{v_1, v_2\}$ and no edges. By minimal fairness, $v_1 \simeq_G^F v_2$. Now assume a graph $G' = (V, \{(v_1, v_2)\})$ with an added edge between v_1 and v_2. By strong positive response, $v_1 \prec_G^F v_2$. However, by 0-incentive compatibility, $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, which is a contradiction.* □

Proposition 9. *There exists no 0-incentive compatible ranking system that satisfies strong monotonicity.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. Assume a graph G with two vertices $V = \{v_1, v_2\}$ and no edges. As $P_G(v_1) = P_G(v_2)$, by strong monotonicity, $v_1 \simeq_G^F v_2$. Now assume a graph $G' = (V, \{(v_1, v_2)\})$ with an added edge between v_1 and v_2. As $P_{G'}(v_1) \subsetneq P_{G'}(v_2)$, $v_1 \prec_G^F v_2$. However, by 0-incentive compatibility, $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, which is a contradiction.* □

Proposition 10. *There exists no nontrivial strongly incentive compatible minimally fair ranking system..*

Proof. *We will prove that for any $G = (V, E)$, and for any $v_1, v_2 \in V$: $v_1 \preceq_G^F v_2$. The proof is by induction on $|E|$.*

Induction Base: *Assume $E = \emptyset$, and let $v_1, v_2 \in V$ be vertices. By minimal fairness, $v_1 \preceq v_2$.*

Inductive Step: *Assume correctness for $|E| \le n$ and prove for $|E| = n + 1$. Assume for contradiction that for some $v_1, v_2 \in V$: $v_2 \prec v_1$. Let $v \in V$ be a vertex such that $S(v) \neq \emptyset$ (such a vertex exists because $|E| > 0$). Note that $|\{x \in V | v \simeq_G^F x\}| < |V|$, because otherwise $v_1 \preceq_G^F x \preceq_G^F v_2$. Let $E' = E \setminus \{(v, x) | x \in V\}$ and $G' = (V, E')$. By the assumption of induction, $|\{x \in V | v \simeq_{G'}^F x\}| = |V|$. Thus, $|\{x \in V | v \prec_{G'}^F x\}| = 0$. By strong incentive compatibility, $0 \le |\{x \in V | v \prec_G^F x\}| \le |\{x \in V | v \prec_{G'}^F x\}| = 0$, thus $|V| = |\{x \in V | v \simeq_{G'}^F x\}| \le |\{x \in V | v \simeq_G^F x\}| < |V|$ which yields a contradiction.* □

Proposition 11. *There exists no 0-incentive compatible nontrivial minimally fair ranking system that satisfies the strong union condition.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. As the conditions of Lemma 1 are satisfied, let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma. Now let $G_1 = (V \setminus \{v_\top\}, E \setminus \{(v', v_\top) \in E | v' \in V\})$ and let $G_2 = (\{v_\top\}, \emptyset)$. From conditions 3 and 5, $S(v_\top) = \emptyset$ and thus G_1 and G_2 satisfy the conditions of the strong union condition with regard to G. Therefore, $v_\perp \prec_G^F v \Rightarrow v_\perp \prec_{G_1}^F v$, in contradiction to condition 1, because $|E_1| \le |E|$ and $|V_1| < |V|$.* □

7. SOME ILLUMINATING LESSONS

Theorems 1 and 2 teach us some surprising lessons about the implications of various versions of the basic properties.

7.1 Strong incentive compatibility is different than 0-incentive compatibility

We have seen in Proposition 10 that, as one would expect, strong incentive compatibility is impossible when assuming minimal fairness. However, it turns out that when we slightly weaken the requirement of incentive compatibility to cover only the *expected* rank of the agent, Proposition 4 shows us this is possible. This means that the level of incentive compatibility has an effect on the existence of ranking systems. We expect that additional interesting ranking systems will become possible as we require k-incentive compatibility for larger values of k.

7.2 Positive Response is not the same as Monotonicity

The Positive response and Monotonicity properties seem, at a glance, to be very similar, as they both informally require that the more votes an agent has, the higher it is ranked. However, looking more deeply, we see that the Positive Response properties require this behavior to be manifested across graphs, while the Monotonicity properties require that the effect be seen within a single graph.

This leads to interesting facts, such as not being able to nontrivially satisfy both Weak Monotonicity and Weak Positive response with incentive compatibility (Proposition 5), while each of the properties could be satisfied separately (Propositions 4 and 1) . Furthermore, Strong Monotonicity cannot be satisfied at all (Proposition 9) with 0-incentive compatibility, while Strong Positive Response *can* be satisfied even with strong incentive compatibility (Proposition 1).

7.3 The Weak Union property matters

Recall that the weak union property requires that when two disjoint graphs are put together, the subgraphs must still be ranked as before.

This property might seem trivial, but the impossibility results in Theorem1 imply that this property has a part in inducing impossibility. The reason for this is twofold:

- The combination of two graphs adds more options for the agents in both subgraphs to vote for, which in order to preserve incentive compatibility, must all preserve the agent's relative rank in the combined graph.
- The weak union property further implies that the ranking system must not rely on the number of vertices in the graph, and moreover, that the minimal nontrivially ranked graph for a given ranking system must be connected.

8. THE ISOMORPHISM PROPERTY AND FURTHER RESEARCH

Most of the ranking systems we have seen up to now in the possibility proofs took advantage of the names of the vertices to determine the ranking. A natural requirement from a ranking system is that the names assigned to the vertices will not take part in determining the ranking. This is formalized by the isomorphism property.

Definition 12. *A ranking system F satisfies* isomorphism *if for every isomorphism function $\varphi : V_1 \mapsto V_2$, and two isomorphic graphs $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq_{\varphi(G)}^F = \varphi(\preceq_G^F)$.*

It turns out that the ranking system F_4 from the possibility proof for 0-incentive compatibility and weak monotonicity (Proposition 4) satisfies isomorphism as well, and thus there exists an 0-incentive compatible ranking system satisfying isomorphism and weak monotonicity. The existence of 0-incentive compatible ranking systems satisfying isomorphism in conjunction with either the weak union property or the weak positive response is an open question.

Further research is also due for the classification of k-incentive compatible ranking systems for $k > 0$.

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