

First Midterm

(75 minutes open book exam)

	credit	max
Question 1		20
Question 2		20
Question 3		20
Question 4		20
Question 5		20
Total		100

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Question 1. ($20 = 7 + 7 + 6$ points). Consider a standard deck of 52 cards.

- (a) How many different five-card hands are possible?
- (b) Of all the possible hands, how many are a full house? (Two cards of one face value, and three of another).
- (c) How many of the hands have only one suit?

Solution. (a) The number of hands is the number of subsets of five cards, which is

$$\begin{aligned}\binom{52}{5} &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 13 \cdot 51 \cdot 5 \cdot 49 \cdot 16 \\ &= 2,598,960.\end{aligned}$$

(b) The number of ways to have a full house with two kings and three queens is $\binom{4}{2} \cdot \binom{4}{3} = 24$. We can choose one of thirteen face values for the pair and one of the remaining twelve for the triplet. The total number of ways to have a full house is therefore

$$24 \cdot 13 \cdot 12 = 3,744.$$

(c) The number of hands with five hearts is $\binom{13}{5} = 1,287$. There are four different suits, hence the number of hands with five cards all the same suit is 5,148.

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Question 2. (20 points). How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where each x_i is an integer larger than -1 ?

Solution. The number of solutions is the same as the number of ways to write

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 29,$$

where each y_i is a non-negative integer. Using five dividers for the twenty-nine copies of 1, we get

$$\begin{aligned} \binom{34}{5} &= \frac{34!}{29!5!} \\ &= \frac{34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 34 \cdot 33 \cdot 8 \cdot 31 \\ &= 278,256 \end{aligned}$$

different sequences, each corresponding to a different way of writing 29.

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Question 3. (20 = 10 + 10 points). Prove or Disprove that the following are equivalence relations.

- (a) Let $x, y \in \mathbb{R}$. We say $x \sim y$ if $x \leq y$.
- (b) Let $x, y \in \mathbb{R}$. We say $x \sim y$ if $\lfloor x \rfloor = \lfloor y \rfloor$.

Solution. (a) The less than relation is not an equivalence relation because it is not symmetric: $1 \leq 2$ but not $2 \leq 1$.

(b) The floor function is an equivalence relation because it is reflexive, symmetric, and transitive. All three properties are easy to check, since $\lfloor x \rfloor$ is an integer and the set of integers under equality is symmetric, transitive, and reflexive.

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Question 4. (20 points). Show that if p is a positive prime number then $\binom{p}{r}$ is an integer multiple of p for every integer $0 < r < p$.

Solution. Recall that

$$\binom{p}{r} = p \cdot \frac{(p-1) \cdot \dots \cdot (p-r+1)}{1 \cdot \dots \cdot r}.$$

We know that $\binom{p}{r}$ is an integer, but none of the integers 1 to r in the denominator divides p . So they divide the product of the other integers in the numerator, that is,

$$\frac{(p-1) \cdot (p-2) \cdot \dots \cdot (p-r+1)}{1 \cdot 2 \cdot \dots \cdot r}$$

is an integer. It follows that $\binom{p}{r}$ is an integer multiple of p .

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Solution. We run the extended gcd algorithm for $j = 144$ and $k = 233$.

Question 5. (20 points). Find the greatest common divisor of 55 and 89 as well as the multiplicative inverse of 55 modulo 89.

j	k	q	r	g	x	y
55	89	1	34	1	34	-21
34	55	1	21	1	-21	13
21	34	1	13	1	13	-8
13	21	1	8	1	-8	5
8	13	1	5	1	5	-3
5	8	1	3	1	-3	2
3	5	1	2	1	2	-1
2	3	1	1	1	-1	1
1	2	2	0	1	1	0

We see that $\gcd(55, 89) = 1$. The multiplicative inverse of 55 modulo 89 is $x \bmod 89 = 34$.