# Second Midterm

(75 minutes open book exam)

| | credit | max |
|---|---|---|
| Question 1 | | 20 |
| Question 2 | | 20 |
| Question 3 | | 20 |
| Question 4 | | 20 |
| Question 5 | | 20 |
| Total | | 100 |

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**Question 1.** (20 points). Suppose that $e_0, e_1, \ldots$ is a sequence of integers defined by $e_0 = 1$, $e_1 = 2$, $e_2 = 3$, and $e_k = e_{k-1} + e_{k-2} + e_{k-3}$ for $k \geq 3$. Prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

**Solution.** We use the strong form of Mathematical Induction.

1. *Base Case.* We have $e_0 \leq 3^0$, $e_1 \leq 3^1$, and $e_2 \leq 3^2$.
2. *Inductive Hypothesis.* Suppose $e_k \leq 3^k$ for all integers $k < n$.
3. *Inductive Step.* We have

$$\begin{aligned} e_n &= e_{n-1} + e_{n-2} + e_{n-3} \\ &\leq 3^{n-1} + 3^{n-2} + 3^{n-3} \\ &\leq 3 \cdot 3^{n-1} \\ &= 3^n. \end{aligned}$$

4. *Inductive Conclusion.* We have $e_n \leq 3^n$ for all nonnegative integers $n$.

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**Question 2.** (20 points). Use a truth table to show that $\neg(p \vee q) \vee \neg(p \vee \neg q)$ is equivalent to $\neg p$. Then, prove that this is true using De Morgan's Law.

**Solution.** The truth table that shows the equivalence of $\neg(p \vee q) \vee \neg(p \vee \neg q)$ and $\neg p$ given below.

| $p$ | $q$ | $\neg(p \vee q)$ | $\vee$ | $\neg(p \vee \neg q)$ | $\neg p$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | F | F | F |
| F | T | F | T | T | T |
| F | F | T | T | F | T |

Table 1: Truth table.

Next we prove the equivalence using De Morgan's Law. Writing $x = \neg(p \vee q) \vee \neg(p \vee \neg q)$, we get

$$\begin{aligned} x &\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \\ &\Leftrightarrow (\neg p \vee \neg p) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q) \wedge (\neg q \vee q) \\ &\Leftrightarrow \neg p. \end{aligned}$$

To get the last line, we observe that $\neg q \vee q$ is always true and can therefore be dropped. Furthermore, for $(\neg p \vee q) \wedge (\neg p \vee \neg q)$ to be true, it must be that $\neg p$ is true, else one of the two terms would be false.

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**Question 3.** (20 points). Recall Pascal's Relation, that is, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. Show that

$$\binom{2m}{m+1} + \binom{2m}{m} = \binom{2m+2}{m+1}/2,$$

for every non-negative integer $m$.

**Solution.** Set $n = 2m$ and $k = m+1$ and reorder the terms in Pascal's Relation from back to front to get

$$\binom{2m}{m+1} + \binom{2m}{m} \quad = \quad \binom{2m+1}{m+1}.$$

The right hand side can be changed to

$$\begin{aligned} \binom{2m+1}{m+1} &= \frac{(2m+1)!}{(m+1)!m!} \\ &= \frac{(2m+2)!}{(m+1)!m!(2m+2)} \\ &= \frac{(2m+2)!}{(m+1)!(m+1)!2} \\ &= \binom{2m+2}{m+1}/2. \end{aligned}$$

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**Question 4.** (20 points). For all integers $n$, let $T(n)$ be the number of binary strings of length $n$ that contain the substring 000. For example, for $n = 4$, we have the strings 1000, 0001, and 0000. All other strings of length four do not contain a substring of three consecutive zeros. Thus, $T(4) = 3$. Write a recurrence relation for $T(n)$.

**Solution a.** To derive a recurrence relation, we define $U(n) = 2^n - T(n)$, the number of binary strings of length $n$ that do not contain 000 as a substring. To get a string of length $n + 1$ that contains 000, we either get the three zeros already in the first $n$ positions or not. The former cases are counted by $2T(n)$, because we can add a zero or a one at the end. The latter cases that are not already counted by $2T(n)$ are counted by $U(n - 3)$. To see this, take a string of length $n - 3$ that does not contain 000 and add 100 at the end. Only if this string ends with 100 can we add another zero to get a string that does contain 000. Hence,

$$\begin{aligned} T(n+1) &= 2T(n) + U(n-3) \\ &= 2T(n) - T(n-3) + 2^{n-3}. \end{aligned}$$

To get this recurrence relation going, we note that $T(n) = 0$ for all $n \leq 2$ and $T(3) = 1$. Then $T(4) = 2T(3) - T(0) + 1 = 3$, which is correct.

Just out of curiosity, we compute $T(5) = 2T(4) - T(1) + 2 = 8$. The corresponding strings are 00000, 00001, 00010, 00011, 10000, 10001, 01000, 11000.

**Solution b.** There is another way that we can think of this recurrence. Suppose we have a binary string of length $n + 1$. We will let the $i^{\text{th}}$ bit be labeled $b_i$. We use the tree in Figure 1 to help us count the number of strings of length $n + 1$ that contain the substring 000. There are $T(n)$ ways to have three consecutive zeros if $b_{n+1}$ is one. Now, we must count the number of ways to have three consecutive zeros if $b_{n+1} = 0$. Well, we must consider two cases. First, if $b_n = 1$, then there are $T(n-1)$ ways to have three consecutive zeros since the three zeros must be in the first $n - 1$ digits. However, if the last two digits are zero, we once again must split into two cases. if $b_{n-2} = 1$, then there are $T(n - 2)$ ways to obtain three consecutive zeros, but if $b_{n-1}b_nb_{n+1} = 000$, then all $2^{n-2}$ strings of this form contain three consecutive zeros. Now, we sum

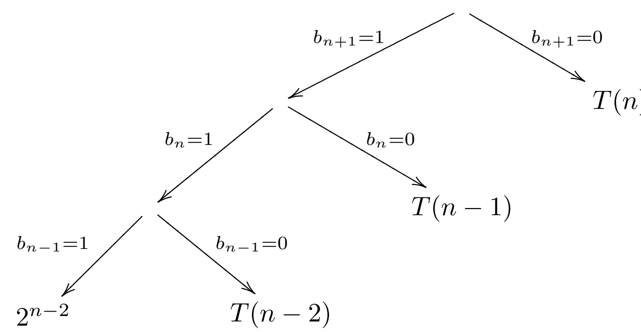


Figure 1: We sum the leaves of this tree to get $T(n + 1)$.

all of the possible (disjoint) ways to get strings containing three consecutive zeros:

$$T(n+1) = T(n) + T(n-1) + T(n-2) + 2^{n-2}$$

Thus, we have our recurrence relation with the initial condition $T(n) = 0$ for all $n \leq 2$.

We note that the two recurrences found are in fact the same since $T(n) = T(n-1)+T(n-2)+T(n-3)+2^{n-3}$ for all $n \geq 0$ implies that

$$2T(n)-T(n-3)+2^{n-3} = T(n)+T(n-1)+T(n-2)+2^{n-2}.$$

The right hand side of the last equation is $T(n+1)$. Thus, we have $T(n+1) = 2T(n) - T(n-3) + 2^{n-3}$.

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**Question 5.** (20 points). Let $n$ be a positive integer. Prove that $\sqrt{n}$ is irrational whenever $n$ is not the square of another integer.

**Solution.** Let $n$ be an integer that is not the square of another integer and assume $\sqrt{n}$ is rational, that is, there are integers $i$ and $j$ such that $\sqrt{n} = \frac{i}{j}$. Then $n = \frac{i^2}{j^2}$ or, equivalently,

$$nj^2 \quad = \quad i^2.$$

The prime factors of $i^2$ are the prime factors of $i$ twice. It follows that the right hand side of the equation has each prime factor an even number of times. Similarly, the decomposition of $j^2$ gives each prime factor an even number of times. For the equation to hold, the decomposition of $n$ into prime factors must give each factor an even number of times. But if this is the case then $n = k^2$ for another integer $k$. Contradiction.