

# Final Exam

(3 hour open book exam)

	credit	max
Question 1		20
Question 2		20
Question 3		20
Question 4		20
Question 5		20
Question 6		20
Question 7		20
Question 8		20
Total		160

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**Question 1.** (20 = 10 + 10 points). Suppose we roll a die. Let  $X$  be the number of dots that appear on the top of the die.

- (a) What is expected value of  $X$ ?
- (b) What is the variance?

**Solution.**

- (a) The expected value is  $E(X) = \frac{1}{6} \sum_{i=1}^6 i$ . This is  $\frac{\binom{7}{2}}{6} = \frac{7}{2}$ .
- (b) The variance of  $X$  is  $V(X) = \frac{1}{6} \sum_{i=1}^6 (i - E(X))^2$ . This is one sixth of  $\frac{5^2}{4} + \frac{3^2}{4} + \frac{1^2}{4} + \frac{1^2}{4} + \frac{3^2}{4} + \frac{5^2}{4}$ , which is equal to  $\frac{35}{12}$ .

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**Question 2.** (20 = 10 + 10 points). The average height of an adult male is 66 inches with a standard deviation of 2 inches. On the other hand, the average basketball player is 78 inches with a standard deviation of 3 inches. Assume both distributions are normal.

- (a) What is the probability that a randomly selected adult male is between 62 and 70 inches?
- (b) What is the probability that a randomly selected basketball player is between 75 and 81 inches?

**Solution.**

- (a) In a normal distribution, the percentage of males within two standard deviations from the mean is 95.5%.
- (b) In a normal distribution, the percentage of males within one standard deviation from the mean is 68%.

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**Question 3.** (20 points). The  $k$ -dimensional hypercube has as its vertices the ordered  $k$ -tuples of zeros and ones. Two vertices are connected by an edge if they differ in exactly one position. For what values of  $k$  is the hypercube Eulerian?

**Solution.** A graph is Eulerian iff it is connected and every vertex has even degree. The  $k$ -dimensional hypercube is connected and every vertex has degree equal to  $k$ . Hence, the hypercube is Eulerian iff  $k$  is even.

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**Question 4.** (20 = 10 + 10 points). Consider the two graphs below. For each graph, determine if it is a planar graph. If it is planar, draw the embedding. If it is not, prove that it is not planar.

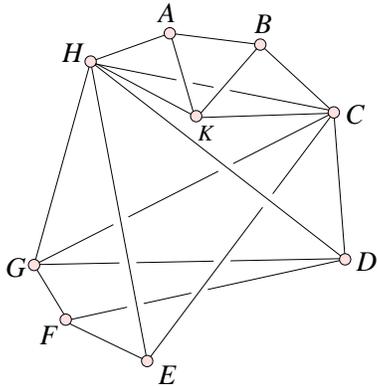


Figure 1: Is this graph planar?

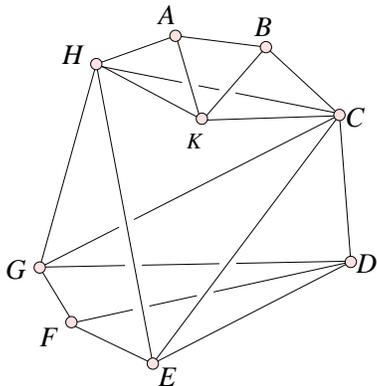


Figure 2: Is this graph planar?

**Solution.** The graph in Figure 1 is not planar. To see this, note that we have a complete bipartite graph of three plus three vertices, namely  $C, F, H$  are connected by three edges edge to  $D, E, G$ . (Alternatively, we could contract the edge  $EF$  and get subgraph that is a complete graph of five vertices.)

The graph in Figure 2 is planar. To see this, move vertex  $K$  above the edge  $CH$  and draw the edges  $CE$  and  $EH$  curved on the outside to avoid the crossings we currently have.

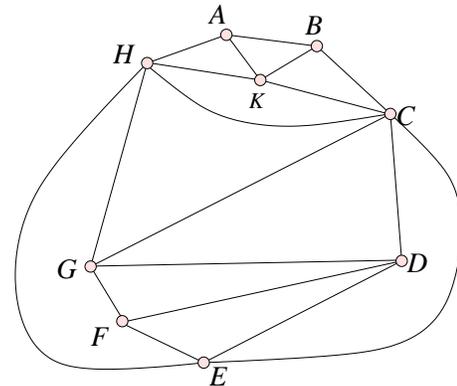


Figure 3: This graph is planar.

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**Question 5.** (20 = 5 + 5 + 5 + 5 points). Let  $p$ ,  $q$ ,  $r$  be the following propositions:

$p$ : "You obtain a lottery ticket."

$q$ : "You win the lottery."

$r$ : "You spend \$1 on the lottery ticket."

- (a) Write the following logical statements using  $p$ ,  $q$ ,  $r$  and logical connectives.
1. You obtain a lottery ticket, but you don't spend \$1 on it.
  2. To win the lottery, it is necessary for you to obtain a lottery ticket.
  3. Obtaining a lottery ticket is sufficient to win the lottery.
  4. You win the lottery if and only if you spend \$1 on the ticket.
  5. Not winning the lottery is necessary for not obtaining a lottery ticket.
- (b) Which of the five logical statements are equivalent?
- (c) Which of the five logical statements are converses of each other?
- (d) Which of the five logical statements are contrapositives of each other?

**Solution.**

- (a)
1.  $p \wedge \neg r$ .
  2.  $q \Rightarrow p$ .
  3.  $p \Rightarrow q$ .
  4.  $q \Leftrightarrow r$ .
  5.  $\neg p \Rightarrow \neg q$ .
- (b) Propositions 2 and 5 are equivalent.
- (c) Propositions 2 and 3 are converses of each other. Since 2 and 5 are equivalent, 3 and 5 are also converse of each other.
- (d) Propositions 2 and 5 and contrapositives of each other.

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**Question 6.** (20 points). Let  $F_n$  be the  $n$ -th Fibonacci number, that is,  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for all  $n \geq 2$ . Prove that for every positive integer  $n$ , we have

$$F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}.$$

**Solution.** We use the Principle of Mathematical Induction (weak form).

For  $n = 1$ , we have  $F_1 = F_2$  and therefore  $F_1^2 = F_1 F_2$ .

Now assume the equation holds for  $n$ .

To prove the equation for  $n + 1$ , we add  $F_{n+1}^2$  on both sides. On the right hand side, we get  $F_n F_{n+1} + F_{n+1}^2 = F_{n+1} F_{n+2}$ , as required.

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**Question 7.** (20 points). Determine  $f(n)$  for  $n = 2^k$ , where  $f(1) = 1$  and  $f(n) = 4f(n/2) + n^2$  for  $k \geq 1$ .

**Solution.** Write  $g(k) = f(2^k)$ . Hence,  $g(0) = 1$  and  $g(k) = 2^2 g(k-1) + 2^{2k}$  for each  $k \geq 1$ . Developing this by repeated substitution, we get

$$\begin{aligned} g(k) &= 2^{2k} + 2^{2k} + 2^4 g(k-2) \\ &= 2^{2k} + 2^{2k} + \dots + 2^{2k} g(0) \\ &= 2^{2k} (k+1). \end{aligned}$$

Hence,  $f(n) = n^2(1 + \log_2 n)$ .

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**Question 8.** (20 points). Let  $G = (V, E)$  and  $H = (V', E')$  be two graphs. A *graph isomorphism* is a bijection  $f: V \rightarrow V'$  such that  $\{a, b\}$  is an edge of  $G$  iff  $\{f(a), f(b)\}$  is an edge of  $H$ . Knowing this, how many trees with five vertices exist, up to isomorphism?

**Solution.** There are only three trees with five vertices that are pairwise non-isomorphic. Use the maximum degree of a vertex to see this. There is one tree with a degree-4 vertex, namely this vertex connected with an edge to each other vertex. There is also only one tree with a degree-3 vertex, namely this vertex connected with an edge to three other vertices and one of these vertices connected with another edge to the last vertex. There is also only one tree whose vertices all have degree 1 or 2, namely the path (linear list).

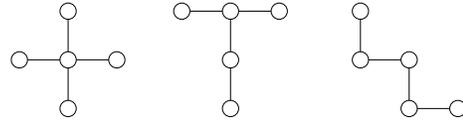


Figure 4: The tree non-isomorphic trees of five vertices.