

3 Equivalence Relations

Equivalence relations are a way to partition a set into subsets of equivalent elements. Being equivalent is then interpreted as being the same, such as different views of the same object or different ordering of the same elements, etc. By counting the equivalence classes, we are able to count the items in the set that are different in an essential way.

Labeling. To begin, we ask how many ways are there to label three of five elements red and the remaining two elements blue? Without loss of generality, we can call our elements A, B, C, D, E . A labeling is an function that associates a color to each element. Suppose we look at a permutation of the five elements and agree to color the first three red and the last two blue. Then the permutation $ABDCE$ would correspond to coloring A, B, D red and C, E blue. However, we get the same labeling with other permutations, namely

$$\begin{array}{lll} ABD; CE & BAD; CE & DAB; CE \\ ABD; EC & BAD; EC & DAB; EC \\ ADB; CE & BDA; CE & DBA; CE \\ ADB; EC & BDA; EC & DBA; EC . \end{array}$$

Indeed, we have $3!2! = 12$ permutations that give the same labeling, simply because there are $3!$ ways to order the red elements and $2!$ ways to order the blue elements. Similarly, every other labeling corresponds to 12 permutations. In total, we have $5! = 120$ permutations of five elements. The set of 120 permutations can thus be partitioned into $\frac{120}{12} = 10$ blocks such that any two permutations in the same block give the same labeling. Any two permutations from different blocks give different labelings, which implies that the number of different labelings is 10. More generally, the number of ways we can label k of n elements red and the remaining $n - k$ elements blue is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$. This is also the number of k -element subsets of a set of n elements.

Now suppose we have three labels, red, green, and blue. We count the number of different labelings by dividing the total number of orderings by the orderings within in the color classes. There are $n!$ permutations of the n elements. We want i elements red, j elements blue, and $k = n - i - j$ elements green. We agree that a permutation corresponding to the labeling we get by coloring the first i elements red, the next j elements blue, and the last k elements green. The number of repeated labelings is thus $i!$ times $j!$ times $k!$ and we have $\frac{n!}{i!j!k!}$ different labelings.

Equivalence relations. We now formalize the above method of counting. A *relation* on a set S is a collection R of ordered pairs, (x, y) . We write $x \sim y$ if the pair (x, y) is in R . We say that a relation is

- *reflexive* if $x \sim x$ for all $x \in S$;
- *symmetric* if $x \sim y$ implies $y \sim x$;
- *transitive* if $x \sim y$ and $y \sim z$ imply $x \sim z$.

We say that the relation is an *equivalence relation* if R is reflexive, symmetric, and transitive. If S is a set and R an equivalence relation on S , then the *equivalence class* of an element $x \in S$ is

$$[x] = \{y \in S \mid y \sim x\}.$$

We note here that if $x \sim y$ then $[x] = [y]$. In the above labeling example, S is the set of permutations of the elements A, B, C, D, E and two permutations are equivalent if they give the same labeling. Recalling that we color the first three elements red and the last two blue, the equivalence classes are $[ABC; DE]$, $[ABD; CE]$, $[ABE; CD]$, $[ACD; BE]$, $[ACE; BD]$, $[ADE; BC]$, $[BCD; AE]$, $[BCE; AD]$, $[BDE; AC]$, $[CDE; AB]$.

Not all relations are equivalence relations. Indeed, there are relations that have none of the above three properties. There are also relations that satisfy any subset of the three properties but none of the rest.

An example: modular arithmetic. We say an integer a is *congruent* to another integer b modulo a positive integer n , denoted as $a \equiv b \pmod{n}$, if $b - a$ is an integer multiple of n . To illustrate this definition, let $n = 3$ and let S be the set of integers from 0 to 11. Then $x \equiv y \pmod{3}$ if x and y both belong to $S_0 = \{0, 3, 6, 9\}$ or both belong to $S_1 = \{1, 4, 7, 10\}$ or both belong to $S_2 = \{2, 5, 8, 11\}$. This can be easily verified by testing each pair. Congruence modulo 3 is in fact an equivalence relation on S . To see this, we show that congruence modulo 3 satisfies the three required properties.

reflexive. Since $x - x = 0 \cdot 3$, we know that $x \equiv x \pmod{3}$.

symmetric. If $x \equiv y \pmod{3}$ then x and y belong to the same subset S_i . Hence, $y \equiv x \pmod{3}$.

transitive. Let $x \equiv y \pmod{3}$ and $y \equiv z \pmod{3}$. Hence x and y belong to the same subset S_i and so do y and z . It follows that x and z belong to the same subset.

More generally, congruence modulo n is an equivalence relation on the integers.

Block decomposition. An equivalence class of elements is sometimes called a *block*. The importance of equivalence relations is based on the fact that the blocks partition the set.

THEOREM. Let R be an equivalence relation on some set S . Then the blocks $S_x = \{y \in S \mid x \sim y, y \in S\}$ for all $x \in S$ partition S .

PROOF. In order to prove that $\bigcup_x S_x = S$, we need to show two things, namely $\bigcup_{x \in S} S_x \subseteq S$ and $\bigcup_{x \in S} S_x \supseteq S$. Each S_x is a subset of S which implies the first inclusion. Furthermore, each $x \in S$ belongs to S_x which implies the second inclusion. Additionally, if $S_x \neq S_y$, then $S_x \cap S_y = \emptyset$ since $z \in S_x$ implies $z \sim x$, which means that $S_x = S_z$, which means that $S_z \neq S_y$. Therefore, z is not related to y , and so $z \notin S_y$. \square

Symmetrically, a partition of S defines an equivalence relation. If the blocks are all of the same size then it is easy to count them.

QUOTIENT PRINCIPLE. If a set S of size p can be partitioned into q classes of size r each, then $p = qr$ or, equivalently, $q = \frac{p}{r}$.

Multisets. The difference between a set and a *multiset* is that the latter may contain the same element multiple times. In other words, a multiset is an unordered collection of elements, possibly with repetitions. We can list the repetitions,

$$\langle\langle c, o, l, o, r \rangle\rangle$$

or we can specify the multiplicities,

$$m(c) = 1, m(o) = 2, m(r) = 1.$$

The *size* of a multiset is the sum of the multiplicities. We show how to count multisets by considering an example, the ways to distribute k (identical) books among n (different) shelves. The number of ways is equal to

- the number of size- k multisets of the n shelves;
- the number of ways to write k as a sum of n non-negative integers.

We count the ways to write k as a sum of n non-negative integers as follows. Choose the first integer of the sum to be p . Now we have reduced the problem to counting the ways to write $k - p$ as the sum of $n - 1$ non-negative integers. For small values of n , we can do this.

For example, let $n = 3$. Then, we have $p + q + r = k$. The choices for p are from 0 to k . Once p is chosen, the choices for q are fewer, namely from 0 to $k - p$. Finally, if p and q are chosen then r is determined, namely $r = k - p - q$. The number of ways to write k as a sum of three non-negative integers is therefore

$$\begin{aligned} \sum_{p=0}^k \sum_{q=0}^{k-p} 1 &= \sum_{p=0}^k (k - p + 1) \\ &= \sum_{p=1}^{k+1} p \\ &= \binom{k+2}{2}. \end{aligned}$$

There is another (simpler) way of finding this solution. Suppose we line up our n books, then place $k - 1$ dividers between them. The number of books between the i -th and the $(i - 1)$ -st dividers is equal to the number of books on the i -th shelf; see Figure 4. We thus have $n + k - 1$ objects, k books plus $n - 1$ dividers. The number of ways to



Figure 4: The above arrangement of books and blocks represents two books placed on the first and last shelves, and one book on the second shelf. As a sum, this figure represents $2 + 1 + 0 + 2$.

choose $n - 1$ dividers from $n + k - 1$ objects is $\binom{n+k-1}{n-1}$. We can easily see that this formula agrees with the result we found for $n = 3$.

Summary. We defined relations and equivalence relations, investigating several examples of both. In particular, modular arithmetic creates equivalence classes of the integers. Finally, we looked at multisets, and saw that counting the number of size- k multisets of n elements is equal to the number of ways to write k as a sum of n non-negative integers.