21 Tours

In this section, we study different ways to traverse a graph. We begin with tours that traverse every edge exactly once and end with tours that visit every vertex exactly once.

Bridges of Königsberg. The Pregel River goes through the city of Königsberg, separating it into two large islands and two pieces of mainland. There are seven bridges connecting the islands with the mainland, as sketched in Figure 27. Is it possible to find a closed walk that traverses each bridge exactly once? We can formalize this question by drawing a graph with four vertices, one for each island and each piece of the mainland. We have an edge for each bridge, as in Figure 27 on the right. The graph has *multi-edges* and is therefore not simple. More generally, we may also allow *loops* that are edges starting and ending at the same vertex. A *Eulerian tour* of such a graph is a closed walk that contains each edge exactly once.

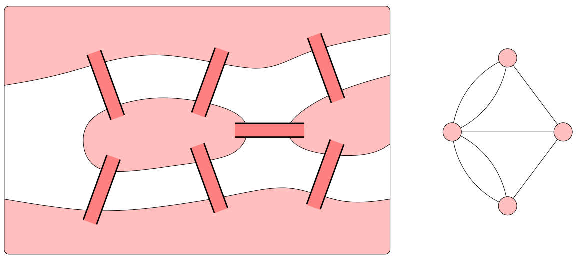

Figure 27: Left: schematic picture of the bridges connecting the islands with the mainland in Königsberg. Right: representation by a graph with four vertices and seven edges.

Eulerian graphs. A graph is *Eulerian* if it permits a Eulerian tour. To decide whether or not a graph is Eulerian, it suffices to look at the local neighborhood of each vertex. The *degree* of a vertex is the number of incident edges. Here we count a loop twice because it touches a vertex at both ends.

EULERIAN TOUR THEOREM. A graph is Eulerian iff it is connected and every vertex has even degree.

PROOF. If a graph is Eulerian then it is connected and each vertex has even degree just because we enter a vertex the same number of times we leave it. The other direction is more difficult to prove. We do it constructively. Given a vertex $u_0 \in V$, we construct a maximal walk, W_0, that leaves each vertex at a yet unused edge. Starting at u_0, the walk continues until we have no more edge to leave the last vertex. Since each vertex has even degree, this last vertex can only be u_0. The walk W_0 is thus necessarily closed. If it is not a Eulerian tour then there are still some unused edges left. Consider a connected component of the graph consisting of these unused edges and the incident vertices. It is connected and every vertex has even degree. Let u_1 be a vertex of this component that also lies on W_0. Construct a closed walk, W_1, starting from u_1. Now concatenate W_0 and W_1 to form a longer closed walk. Repeating this step a finite number of times gives a Eulerian tour. ◻

All four vertices of the graph modeling the seven bridges in Königsberg have odd degree. It follows there is no closed walk that traverses each bridge exactly once.

Hamiltonian graphs. Consider the *pentagon dodecahedron*, the Platonic solid bounded by twelve faces, each a regular pentagon. Drawing the corners as vertices and the sides of the pentagons as edges, we get a graph as in Figure 28. Recall that a cycle in a graph is a closed walk in which no vertex is repeated. A *Hamiltonian cycle* is a closed walk that visits every vertex exactly once. As indicated by the shading of some edges in Figure 28, the graph of the pentagon dodecahedron has a Hamiltonian cycle. A graph is *Hamiltonian* if it permits a Hamiltonian cycle. Deciding whether or not a graph is Hamiltonian turns out to be much more difficult than deciding whether or not it is Eulerian.

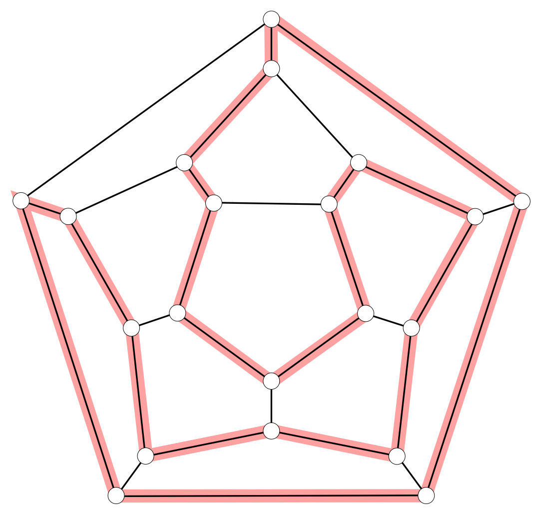

Figure 28: A drawing of a pentagon dodecahedron in which the lengths of the edges are not in scale.

A sufficient condition. The more edges we have, the more likely it is to find a Hamiltonian cycle. It turns out that beyond some number of edges incident to each vertex, there is always a Hamiltonian cycle.

DIRAC'S THEOREM. If G is a simple graph with $n \geq 3$ vertices and each vertex has degree at least $\frac{n}{2}$ then G is Hamiltonian.

PROOF. Assume G has a maximal set of edges without being Hamiltonian. Letting x and y be two vertices not adjacent in G, we thus have a path from x to y that passes through all vertices of the graph. We index the vertices along this path, with $u_1 = x$ and $u_n = y$, as in Figure 29. Now suppose x is adjacent to a vertex u_{i+1}. If y is

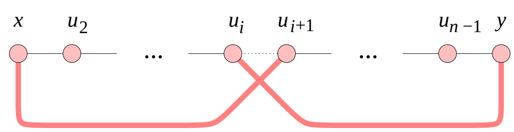

Figure 29: If x is adjacent to u_{i+1} and y is adjacent to u_i then we get a Hamiltonian cycle by adding these two edges to the path and removing the edge connecting u_i to u_{i+1}.

adjacent to u_i then we have a Hamiltonian cycle as shown in Figure 29. Thus, for every neighbor u_{i+1} of x, we have a non-neighbor u_i of y. But x has at least $\frac{n}{2}$ neighbors which implies that y has at least $\frac{n}{2}$ non-neighbors. The degree of y is therefore at most $(n-1) - \frac{n}{2} = \frac{n}{2} - 1$. This contradicts the assumption and thus implies the claim. ◻

The proof of Dirac's Theorem uses a common technique, namely assuming an extreme counterexample and deriving a contradiction from this assumption.

Summary. We have learned about Eulerian graphs which have closed walks traversing each edge exactly once. Such graphs are easily recognized, simply by checking the degree of all the vertices. We have also learned about Hamiltonian graphs which have closed walks visiting each vertex exactly once. Such graphs are difficult to recognize. More specifically, there is no known algorithm that can decide whether a graph of n vertices is Hamiltonian in time that is at most polynomial in n.