23 Planar Graphs

Although we commonly draw a graph in the plane, using tiny circles for the vertices and curves for the edges, a graph is a perfectly abstract concept. We now talk about constraints on graphs necessary to be able to draw a graph in the plane without crossings between the curves. This question forms a bridge between the abstract and the geometric study of graphs.

Drawings and embeddings. Let $G = (V, E)$ be a simple, undirected graph and let $\mathbb{R}^2$ denote the two-dimensional real plane. A *drawing* maps every vertex $u \in V$ to a point $\varepsilon(u)$ in $\mathbb{R}^2$, and it maps every edge $uv \in E$ to a curve with endpoints $\varepsilon(u)$ and $\varepsilon(v)$; see Figure 33. The drawing is an *embedding* if

1. vertices are mapped to distinct points;
2. edge are mapped to curves without self-intersections;
3. a curve does not pass through a point, unless the corresponding edge and vertex are incident, in which case the point is an endpoint of the curve;
4. two curves are disjoint, unless the corresponding edges are incident to a common vertex, in which case the curves share a common endpoint.

Not every graph can be drawn without crossings between the curves. The graph G is *planar* if it has an embedding in the plane.

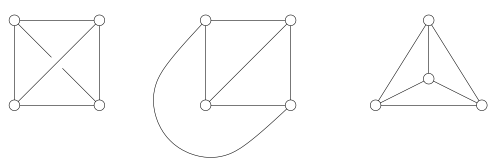

Figure 33: Three drawings of K_4. From left to right a drawing that is not an embedding, an embedding with one curved edge, and a straight-line embedding.

Euler's Formula. Think of the plane as an infinite piece of paper which you cut along the curves with a pair of scissors. Each piece of the paper that remains connected after the cutting is called a *face* of the embedding. We write $n = |V|$, $m = |E|$, and ℓ for the number of faces. Euler's Formula is a linear relation between the three numbers.

EULER'S FORMULA. For an embedding of a connected graph we have $n - m + \ell = 2$.

PROOF. Choose a spanning tree (V, T) of (V, E). It has n vertices, $|T| = n - 1$ edges, and one face. We have $n - (n - 1) + 1 = 2$, which proves the formula if G is a tree. Otherwise, draw the remaining edges, one at a time. Each edge decomposes one face into two. The number of vertices does not change, m increases by one, and ℓ increases by one. Since the graph satisfies the claimed linear relation before drawing the edge it satisfies the relation also after drawing the edge. ◻

We get bounds on the number of edges and faces, in terms of the number of vertices, by considering *maximally connected* graphs for which adding any other edge would violate planarity. Every face of a maximally connected planar graph with three or more vertices is necessarily a triangle, for if there is a face with more than three edges we can add an edge without crossing any other edge. Let $n \geq 3$ be the number of vertices, as before. Since every face has three edges and every edge belong to two triangles, we have $3\ell = 2m$. We use this relation to rewrite Euler's Formula: $n - m + \frac{2m}{3} = 2$ and $n - \frac{3\ell}{2} + \ell = 2$ and therefore $m = 3n - 6$ and $\ell = 2n - 4$. Every planar graph can be completed to a maximally connected planar graph, which implies that it has at most these numbers of edges and faces.

Note that the sum of vertex degrees is twice the number of edges, and therefore $\sum_u \deg(u) \leq 6n - 12$. It follows that every planar graph has a vertex of degree less than six. We will see uses of this observation in coloring planar graphs and in proving that they have straight-line embeddings.

Non-planarity. We can use the consequences of Euler's Formula to prove that the complete graph of five vertices and the complete bipartite graph of three plus three vertices are not planar. Consider first K_5, which is drawn in Figure 34, left. It has $n = 5$ vertices and $m = 10$ edges,

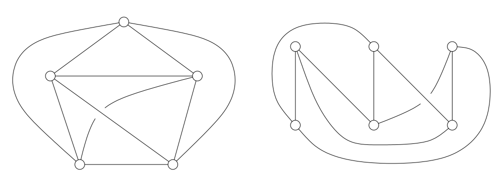

Figure 34: K_5 on the left and $K_{3,3}$ on the right.

contradicting the upper bound of at most $3n - 6 = 9$ edges for maximally connected planar graphs. Consider second $K_{3,3}$, which is drawn in Figure 34, right. It has $n = 6$ vertices and $m = 9$ edges. Each cycle has even length, which implies that each face has four or more edges. We get $4\ell \leq 2m$ and $m \leq 2n - 4 = 8$ after plugging the inequality into Euler's Formula, again a contradiction.

In a sense, K_5 and $K_{3,3}$ are the quintessential non-planar graphs. Two graphs are *homeomorphic* if one can be obtained from the other by a sequence of operations, each deleting a degree-2 vertex and merging its two edges into one or doing the inverse.

KURATOWSKI'S THEOREM. A graph G is planar iff no subgraph of G is homeomorphic to K_5 or to $K_{3,3}$.

The proof of this result is a bit lengthy and omitted. We now turn to two applications of the structural properties of planar graphs we have learned.

Vertex coloring. A *vertex k-coloring* is a map $\chi : V \rightarrow \{1, 2, \ldots, k\}$ such that $\chi(u) \neq \chi(v)$ whenever u and v are adjacent. We call $\chi(u)$ the *color* of the vertex u. For planar graphs, the concept is motivated by coloring countries in a geographic map. We model the problem by replacing each country by a vertex and by drawing an edge between the vertices of neighboring countries. A famous result is that every planar graph has a 4-coloring, but proving this fills the pages of a thick book. Instead, we give a constructive argument for the weaker result that every planar graph has a 5-coloring. If the graph has five or fewer vertices then we color them directly. Else we perform the following four steps:

`Step 1.` Remove a vertex $u \in V$ with degree $k = \deg(u) \leq 5$, together with the k incident edges.

`Step 2.` If $k = 5$ then find two neighbors v and w of the removed vertex u that are not adjacent and merge them into a single vertex.

`Step 3.` Recursively construct a 5-coloring of the smaller graph.

`Step 4.` Add u back into the graph and assign a color that is different from the colors of its neighbors.

Why do we know that vertices v and w in Step 2 exist? To see that five colors suffice, we just need to observe that the at most five neighbors of u use up at most four colors. The idea of removing a small-degree vertex, recursing for the remainder, and adding the vertex back is generally useful. We show that it can also be used to construct embeddings with straight edges.

Convexity and star-convexity. We call a region S in the plane *convex* if for all points $x, y \in S$ the line segment with endpoints x and y is contained in S. Figure 35 shows examples of regions of either kind. We call S *star-convex*

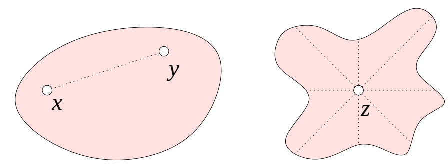

Figure 35: A convex region on the left and a non-convex star-convex region on the right.

if there is a point $z \in S$ such that for every point $x \in S$ the line segment connecting x with z is contained in S. The set of such points z is the *kernel* of S.

It is not too difficult to show that every pentagon is star-convex: decompose the pentagon using two diagonals and choose z close to the common endpoint of these diagonals, as shown in Figure 36. Note however that not every hexagon is star-convex.

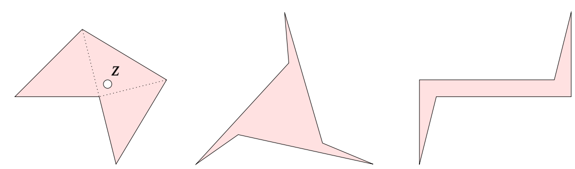

Figure 36: A (necessarily) star-convex pentagon and two non-star-convex hexagons.

Straight-line embedding. A *straight-line embedding* maps every (abstract) edge to the straight line segment connecting the images of its two vertices. We prove that every planar graph has a straight-line embedding using the fact that it has a vertex of degree at most five. To simplify the construction, we assume that the planar graph G is maximally connected and we fix the 'outer' triangle abc. Furthermore, we observe that if G has at least four vertices then it has a vertex of degree at most 5 that is different from a, b and c. Indeed, the combined degree of a, b, c is at least 7. The combined degree of the other $n-3$ vertices is therefore at most $6n - 19$, which implies the average is still less than 6, as required.

`Step 1.` Remove a vertex $u \in V - \{a, b, c\}$ with degree $k = \deg(u) \leq 5$, together with the k incident edges. Add $k-3$ edges to make the graph maximally connected again.

Step 2. Recursively construct a straight-line embedding of the smaller graph.

Step 3. Remove the added $k-3$ edges and map u to a point $\varepsilon(u)$ inside the kernel of the k-gon. Connect $\varepsilon(u)$ with line segments to the vertices of the k-gon.

Figure 37 illustrates the recursive construction. It would be fairly straightforward to turn the construction into a recursive algorithm, but the numerical quality of the embeddings it gives is not great.

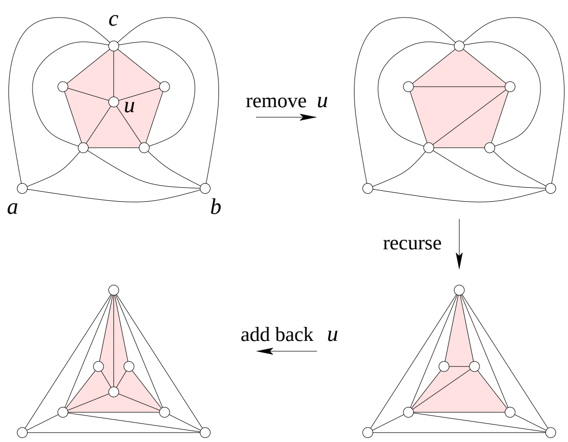

Figure 37: We fix the outer triangle abc, remove the degree-5 vertex u, recursively construct a straight-line embedding of the rest, and finally add the vertex back.