

CPS 173

Kidney exchanges

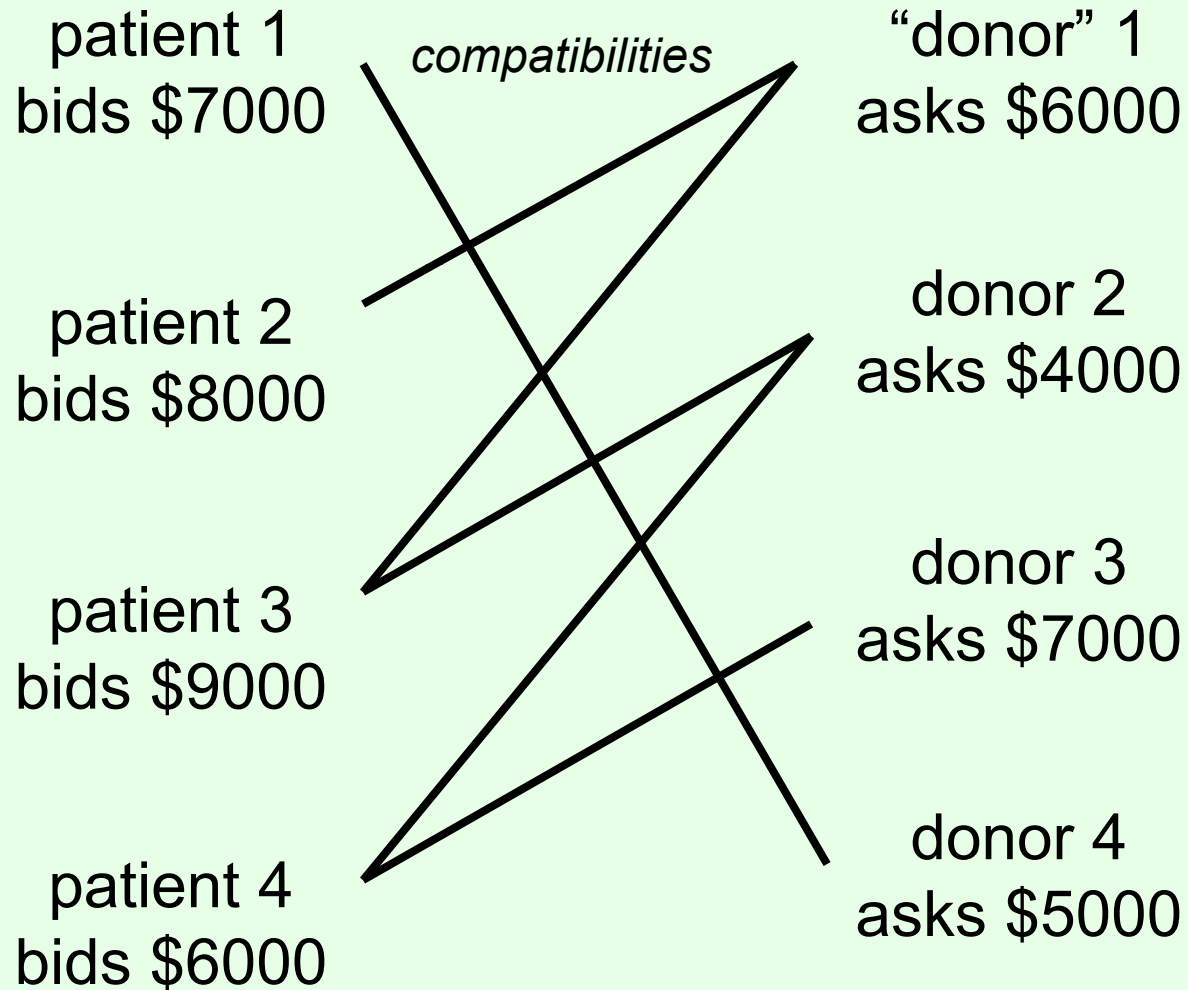
(largely follows Abraham, Blum, Sandholm 2007 paper)

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Kidney transplants

- **Kidneys** filter waste from blood
- Kidney failure results in death in months
- **Dialysis**: regularly get blood filtered in hospital using external machine
 - Low quality of life
- Preferred option: kidney transplant
 - Cadaver kidneys
 - Donation from live patient (better)
- Must be compatible
- Shortage of kidneys...

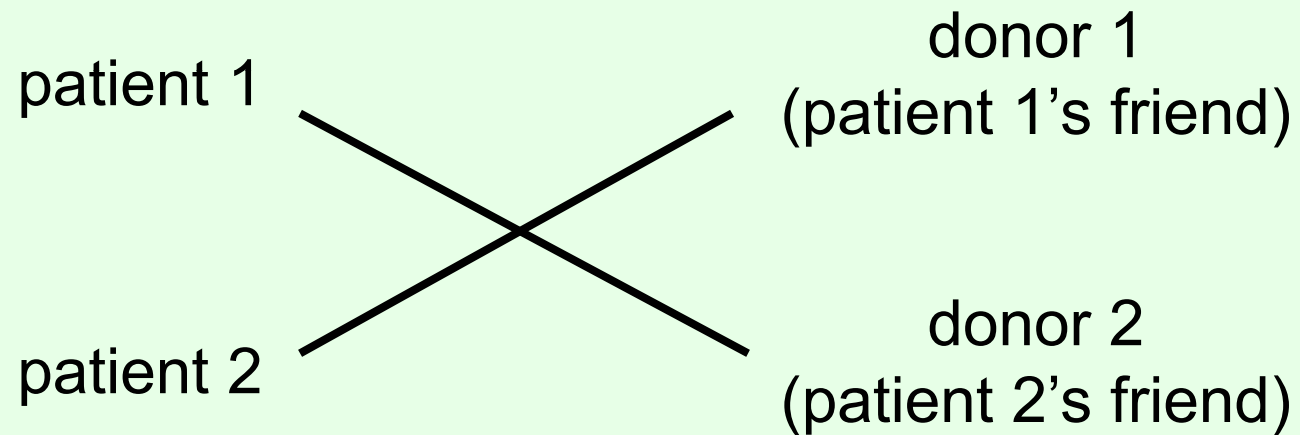
An imaginary kidney exchange with money



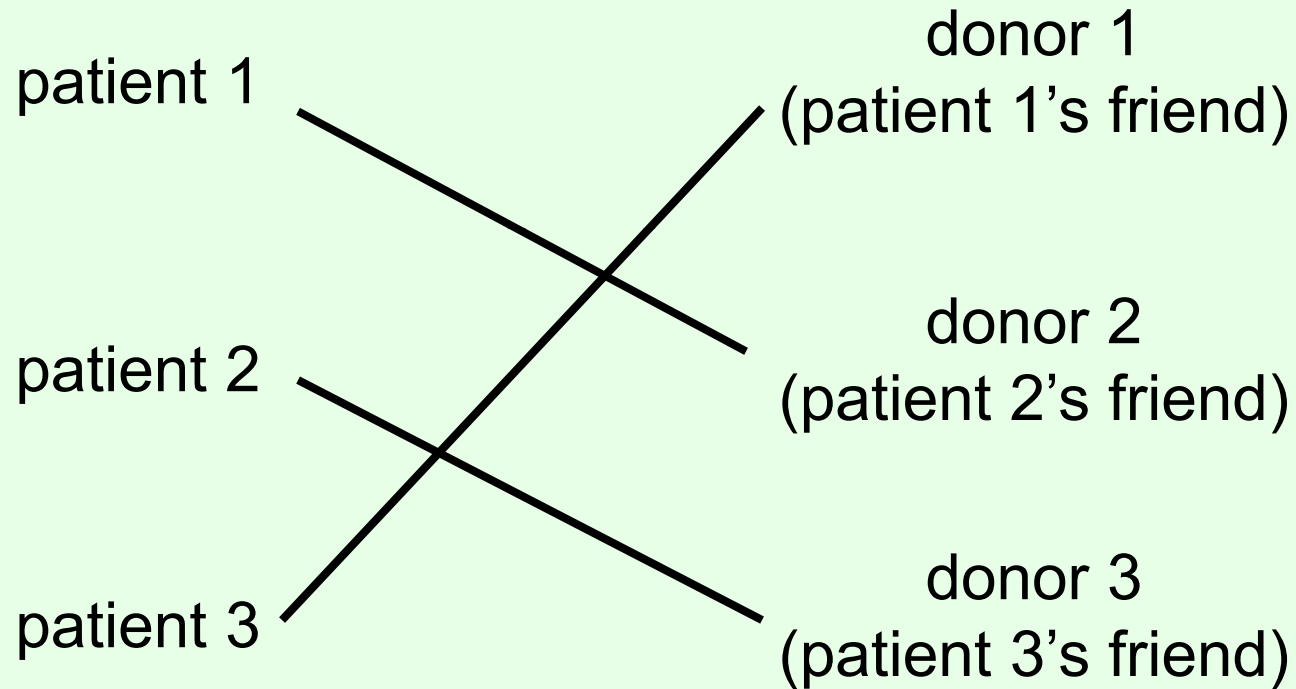
Selling kidneys is illegal!

- Large international black market
 - Desperate people on both ends...
- What can we do legally?

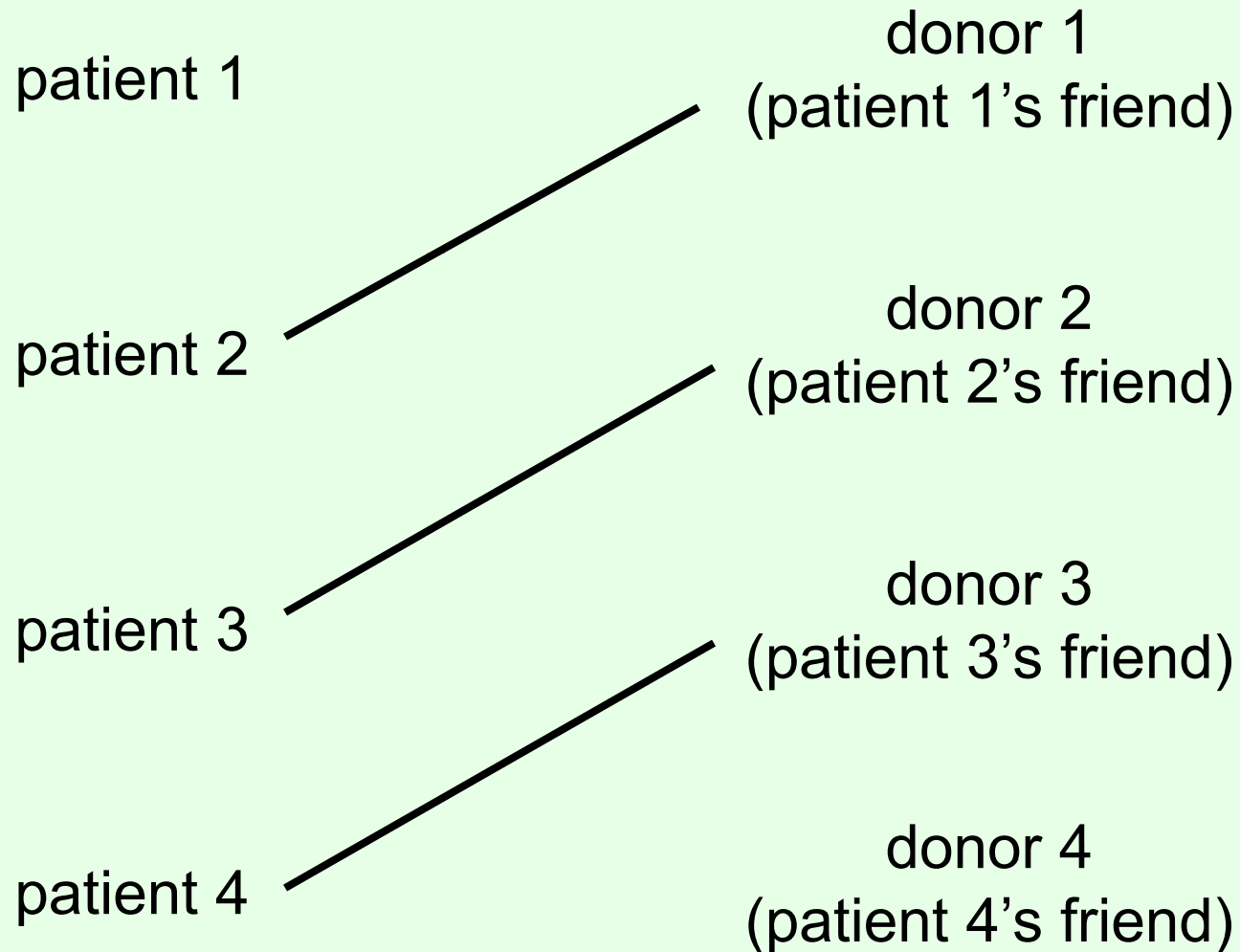
Kidney exchange



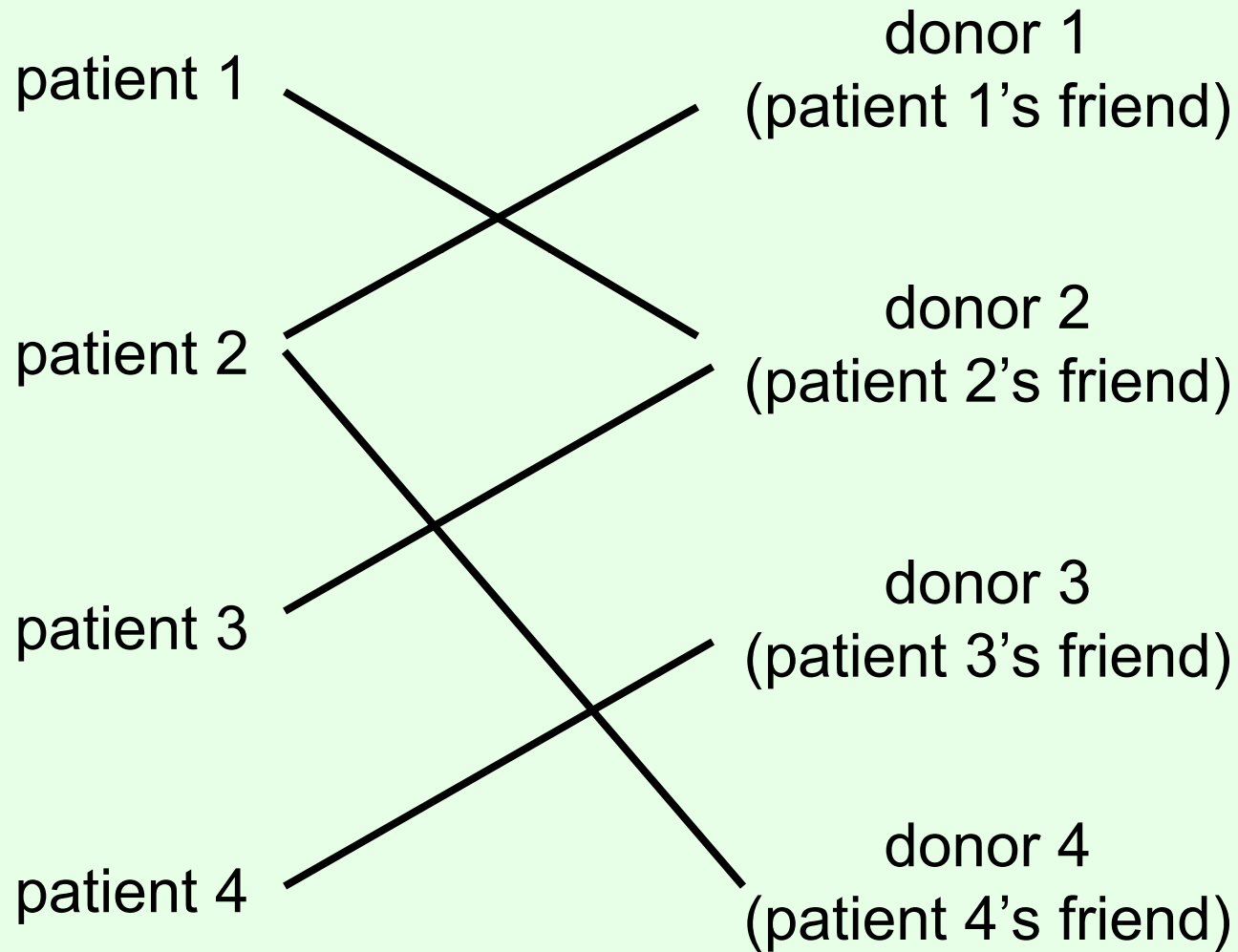
Kidney exchange (3-cycle)



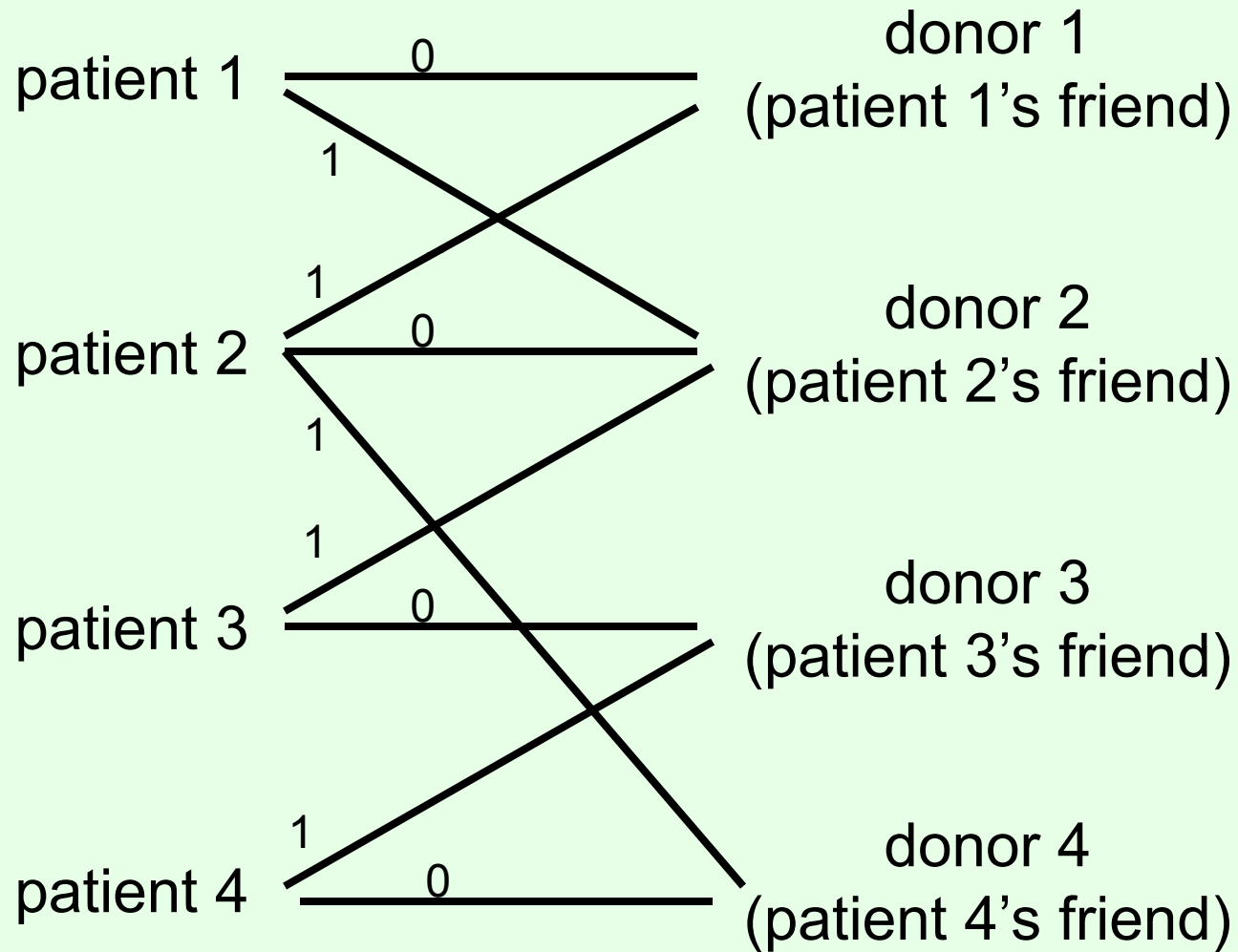
Another example



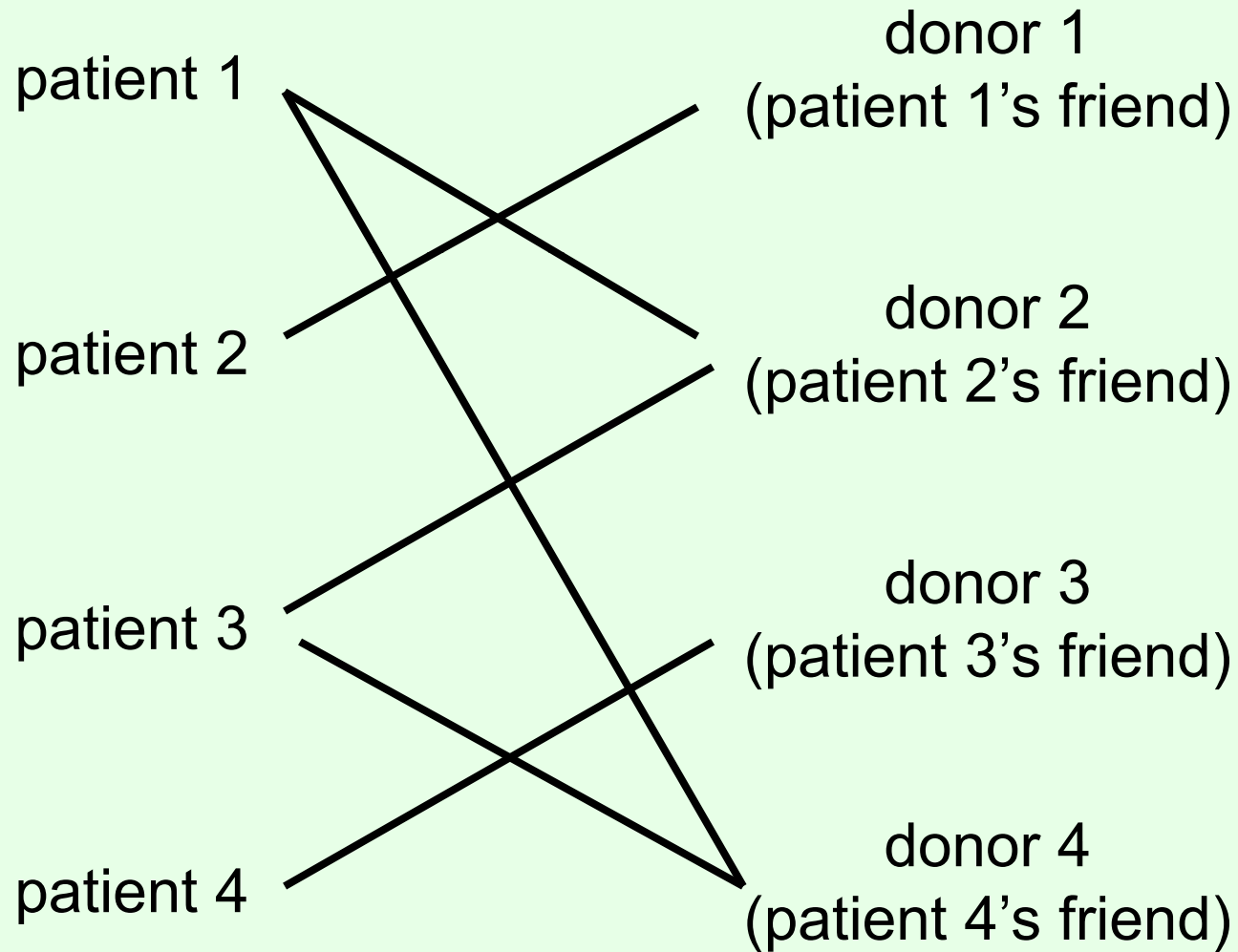
More complex example



Solving kidney exchange as maximum weighted bipartite matching



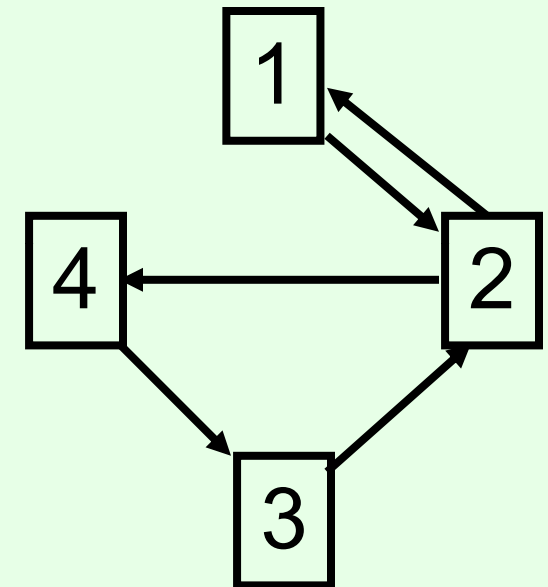
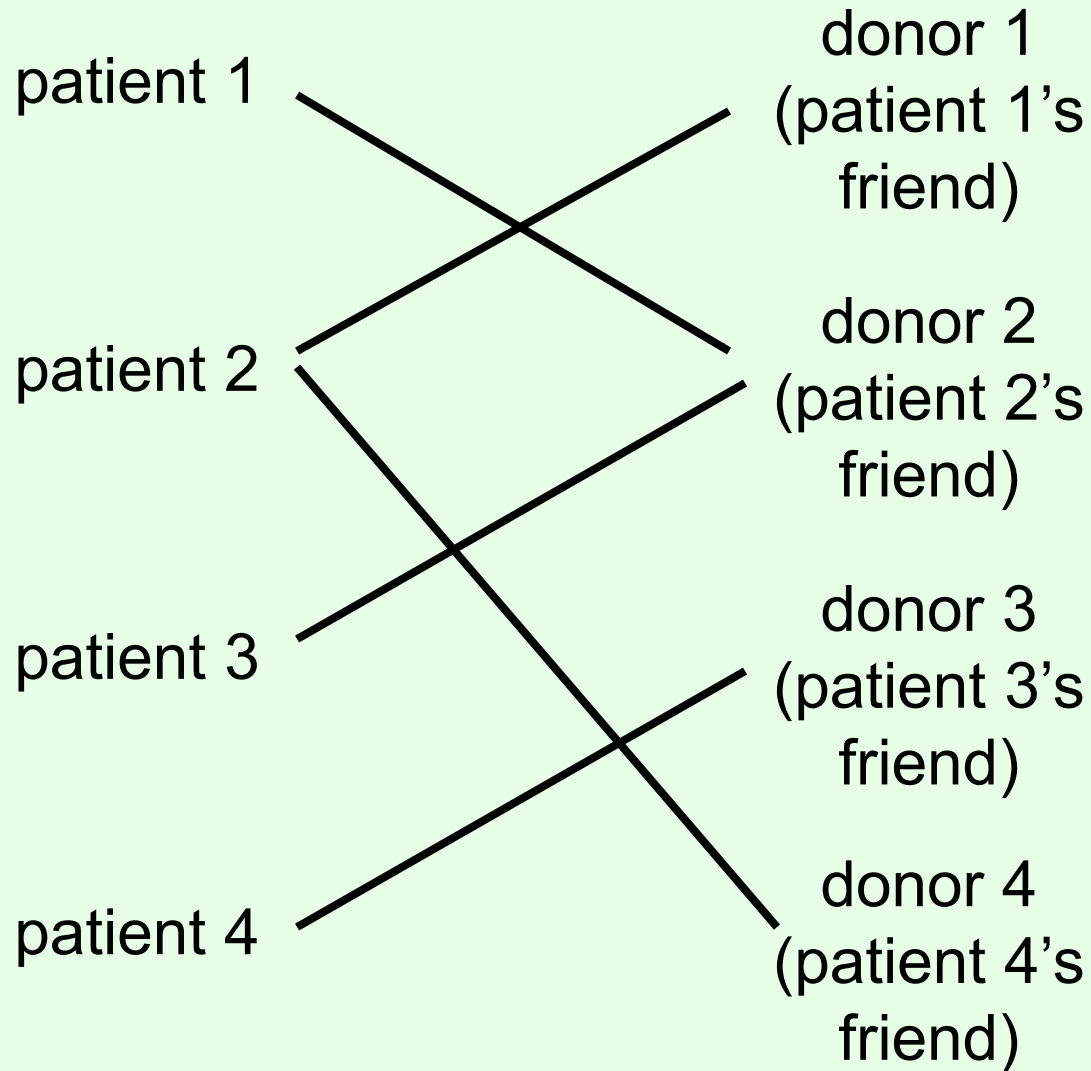
Which solution is better?



Long cycles are impractical

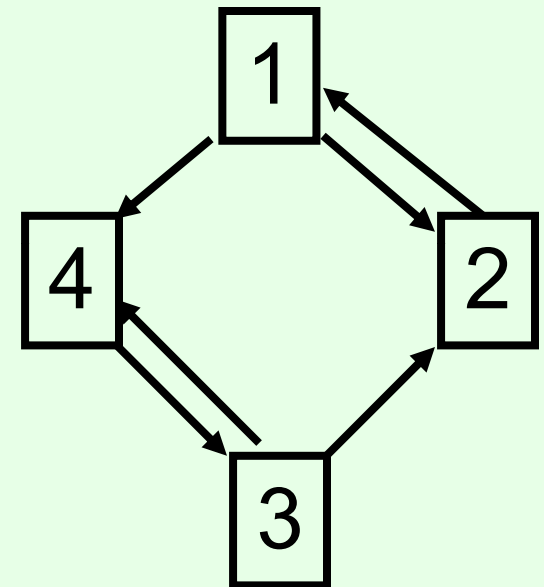
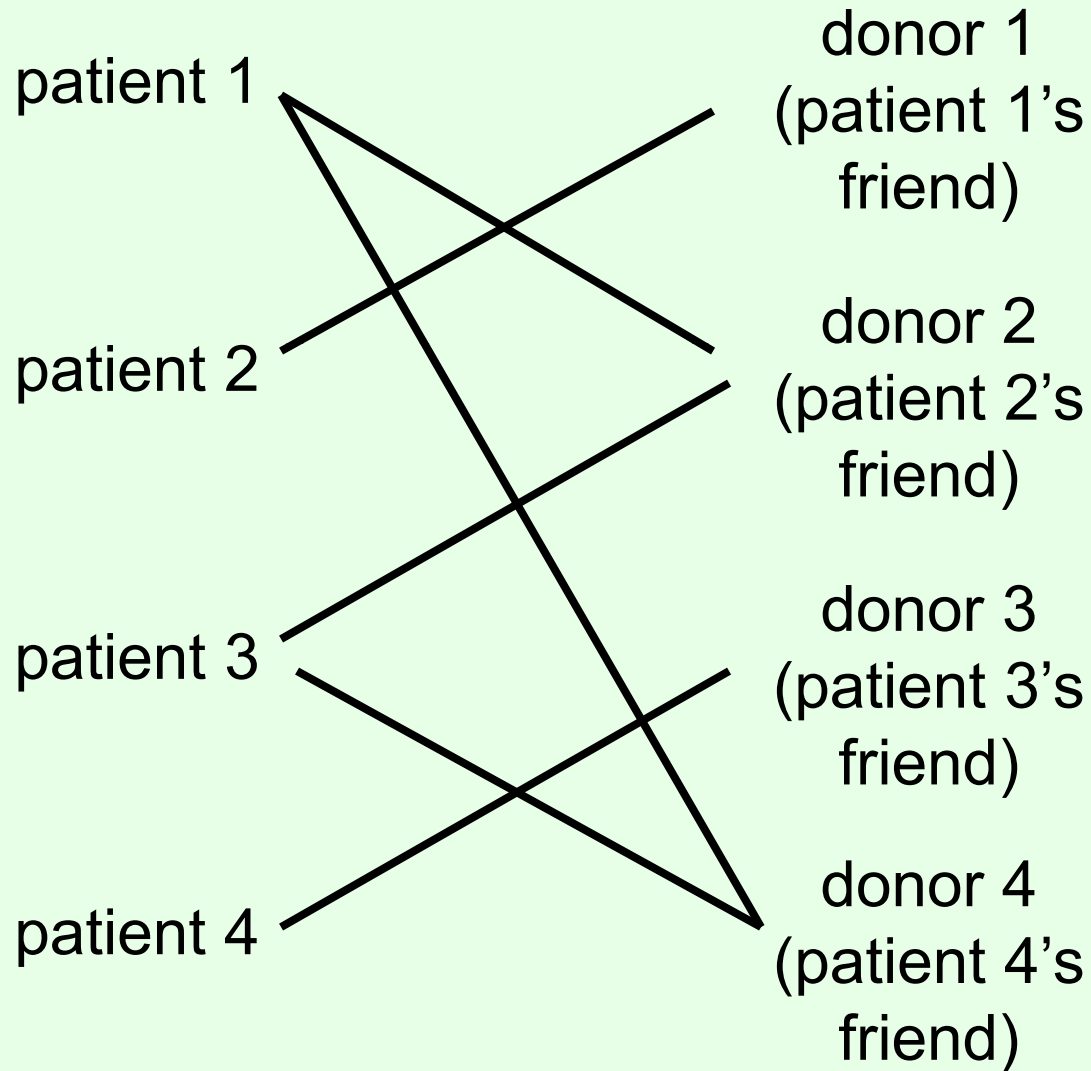
- All patients in a cycle must be operated on simultaneously
 - Otherwise donor can wait for friend to receive kidney, then back out
 - Contracts to donate an organ not binding
- If last-minute test reveals incompatibility, whole thing falls apart
- Require each cycle has length at most k

Different representation



edge from i to j =
patient i wants
donor j 's kidney

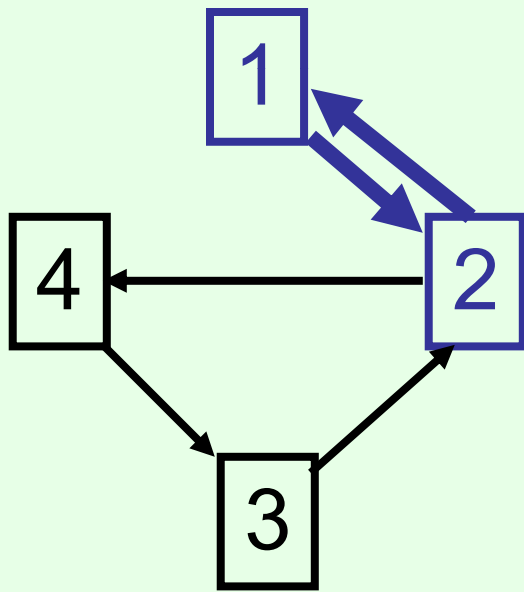
Different representation



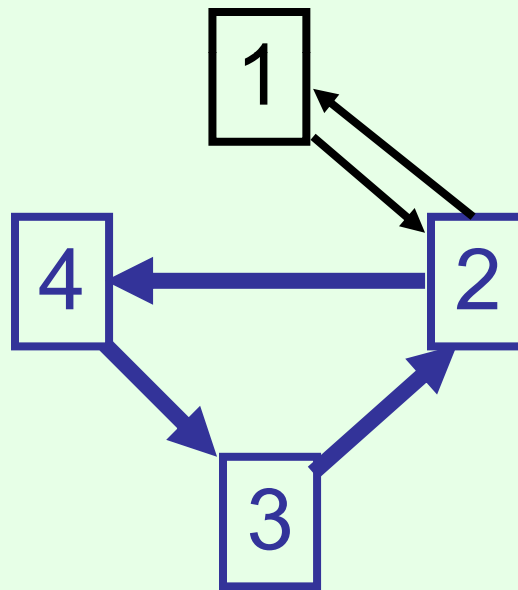
edge from i to j =
patient i wants
donor j 's kidney

Market clearing problem

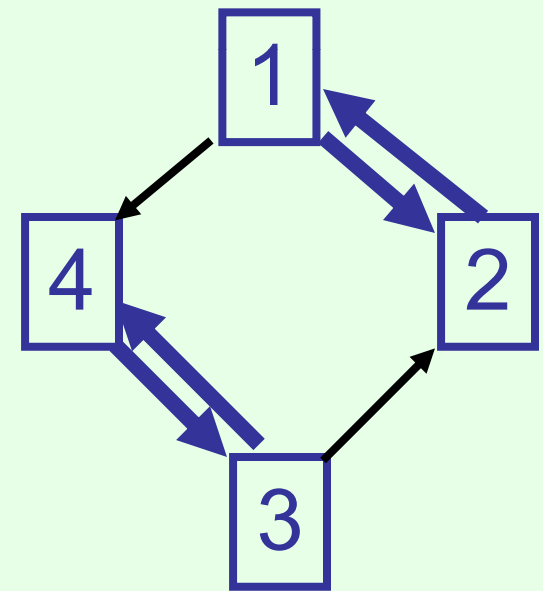
- Try to cover as many vertices as possible with (vertex-)disjoint cycles of length at most k



$k=2$



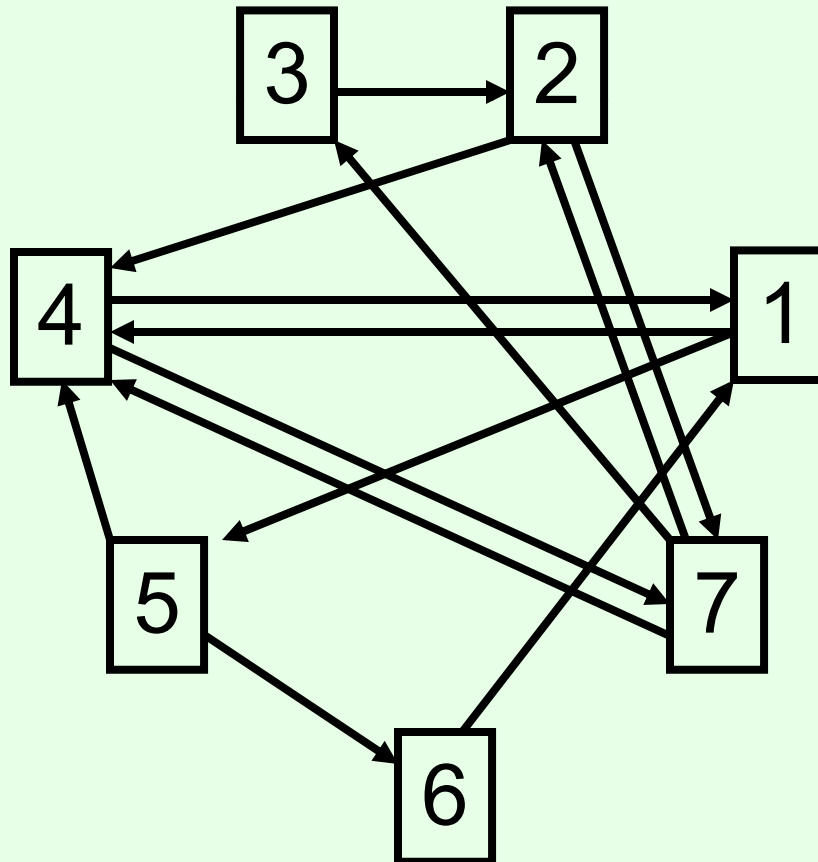
$k=3$



$k=2,3$

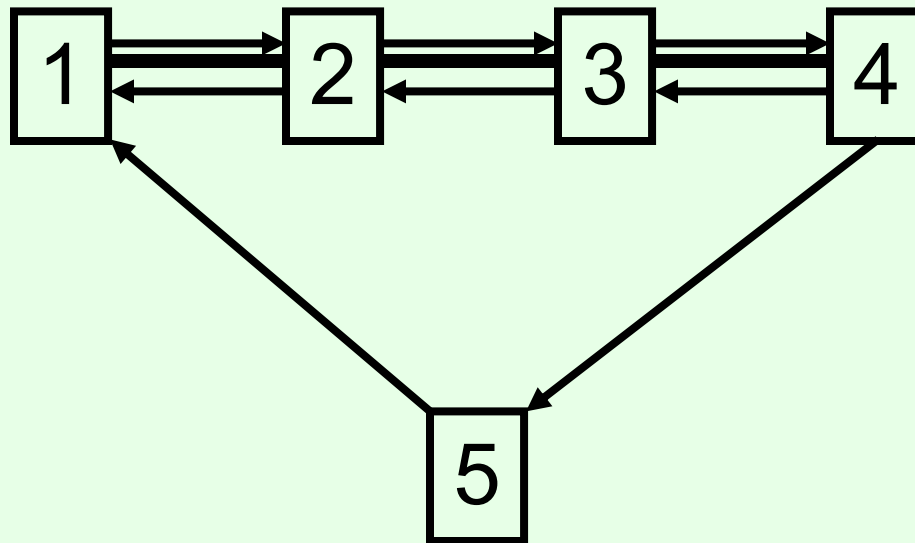
Market clearing problem

- Try to cover as many vertices as possible with (vertex-)disjoint cycles of length at most k



Special case: $k=2$

- If edges go in both directions, replace by undirected edge
- Remove other edges



- Maximum matching problem!

Complexity

- $k = 2$: in P by maximum matching
- $k = \text{number of vertices (no constraint)}$: in P by maximum weighted bipartite matching
- $k = 3, 4, 5, \dots$: NP-hard!

An integer programming formulation

- For each edge from i to j , make a binary variable x_{ij}
 - 1 if i gets j 's kidney, 0 otherwise
- maximize $\sum_{ij} x_{ij}$
- subject to:
- for every i : $\sum_j x_{ij} = \sum_j x_{ji}$
 - (number of kidneys received by i = number of kidneys given by i)
- for every j : $\sum_i x_{ij} \leq 1$
 - (j gives at most 1 kidney)
- for every path $i_1 i_2 \dots i_k i_{k+1}$ with $i_1 \neq i_{k+1}$: $\sum_{1 \leq j \leq k} x_{i_j i_{j+1}} \leq k-1$
 - (no path of length k that doesn't end up where it started, hence no cycles greater than k)

Another integer programming formulation

(turns out better)

- For each cycle c of length at most k , make a binary variable x_c
 - 1 if all edges on this cycle are used, 0 otherwise
- maximize $\sum_c |c| x_c$
- subject to:
- for every vertex i : $\sum_{c: i \text{ in } c} x_c \leq 1$
 - (every vertex in at most one used cycle)

Program size

- Even for small k , number of paths/cycles is too large in reasonably large exchanges
- Solution: generate constraints/variables on the fly during solving
 - Constraint/column generation

Another integer program (not in paper)

- Say an “event” is a set of simultaneous operations
- Denote events by $t = 1, \dots, T$ (how big should T be?)
- For each edge from i to j , for each t , make a binary variable x_{ijt}
 - 1 if i gets j 's kidney in event t , 0 otherwise
- maximize $\sum_{i,j,t} x_{ijt}$
- subject to:
- for every i , t : $\sum_j x_{ijt} = \sum_j x_{jit}$
 - (number of kidneys received by i in event t = number of kidneys given by i in event t)
- for every j : $\sum_{i,t} x_{ijt} \leq 1$
 - (j gives at most 1 kidney overall)
- for every t : $\sum_{i,j} x_{ijt} \leq k$
 - (at most k operations per event)

Other applications

- Barter exchanges: agents want to swap items without paying money
- Peerflix (DVDs)
- Read It Swap It (books)
- Intervac (holiday houses)
- National odd shoe exchange
 - People with different foot sizes
 - Amputees