CompSci 102
Spring 2012

Prof. Rodger

January 11, 2012

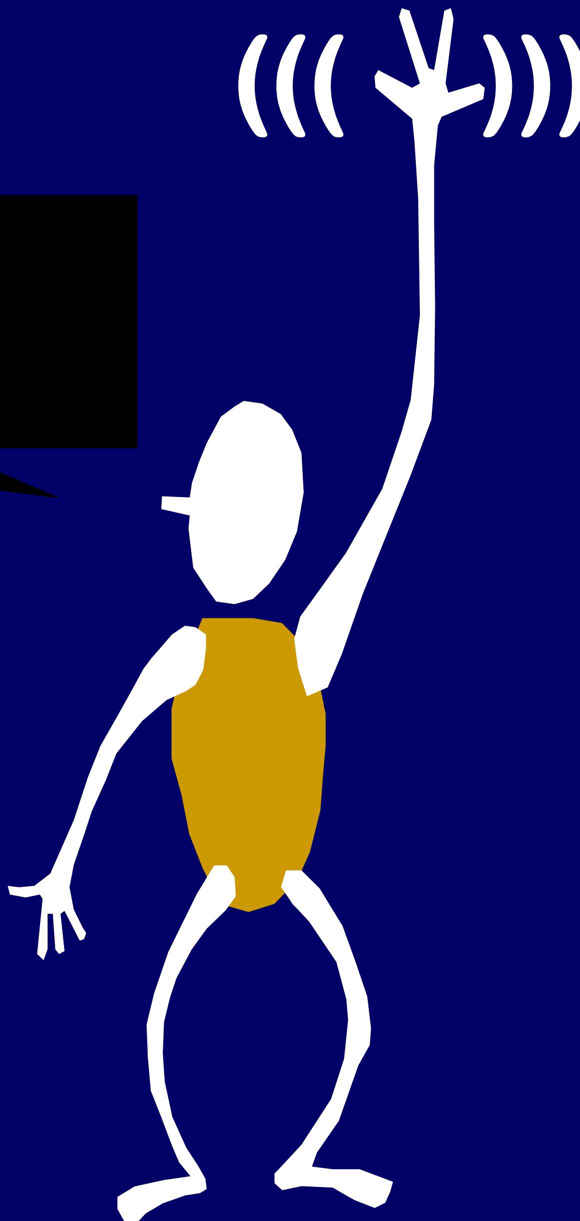
Announcements/Logistics

No recitation Friday the 13th (tomorrow)

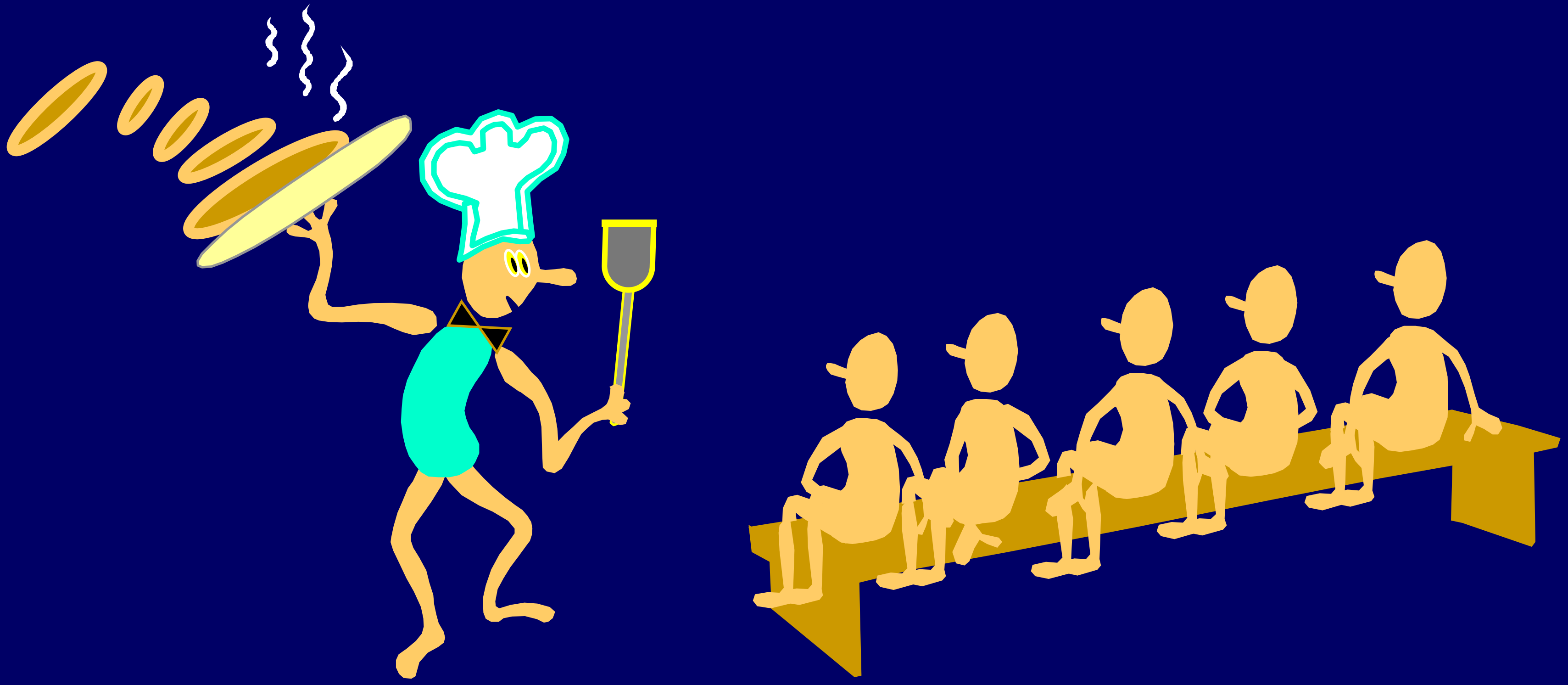
Read the information/policies about this course on the course web site.

www.cs.duke.edu/courses/spring12/cps102

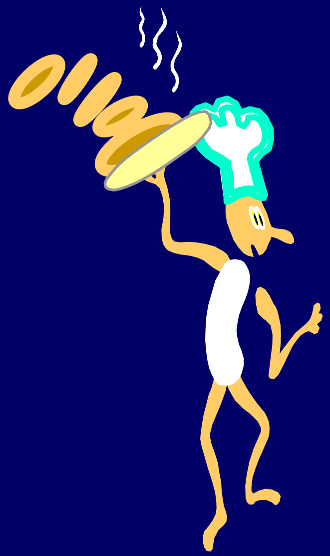
Please feel free
to ask questions!



Pancakes With A Problem!



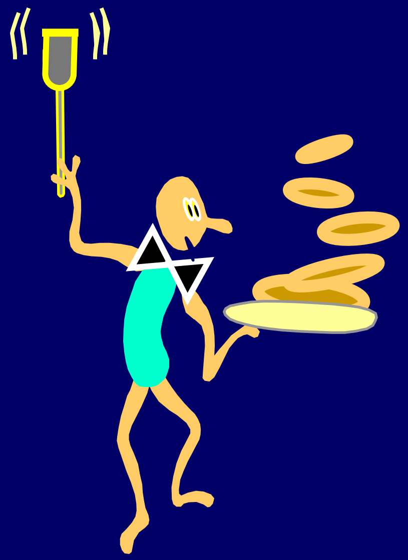
Lecture adapted from Bruce Maggs/Lecture developed at Carnegie Mellon, primarily by Prof. Steven Rudich.



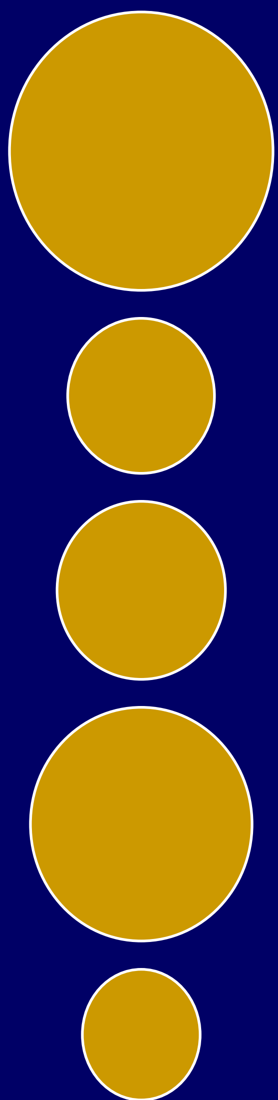
The chef at our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes.

Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom).

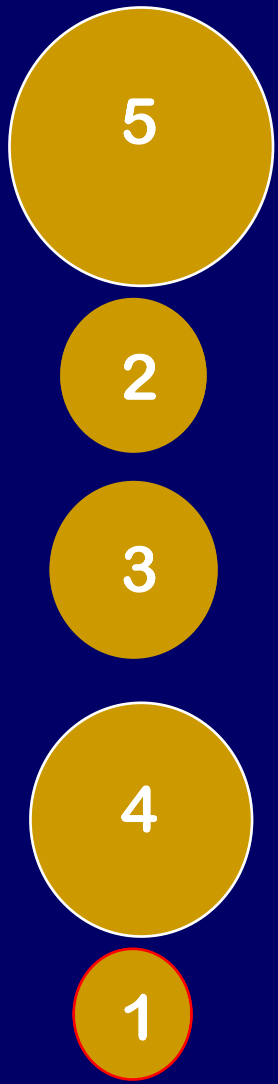
I do this by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary.



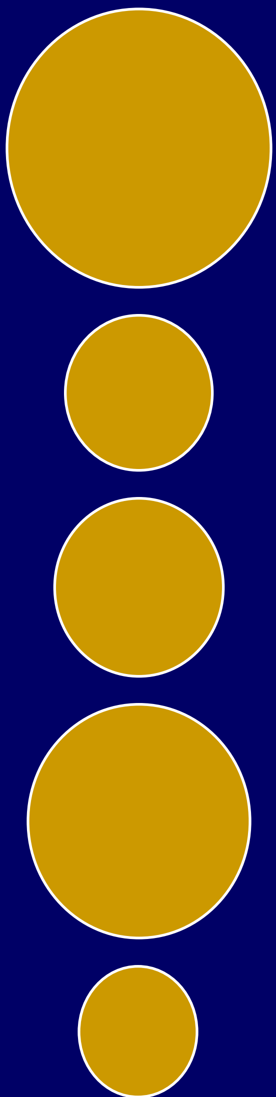
Developing A Notation: Turning pancakes into numbers



Developing A Notation: Turning pancakes into numbers



Developing A Notation: Turning pancakes into numbers



5

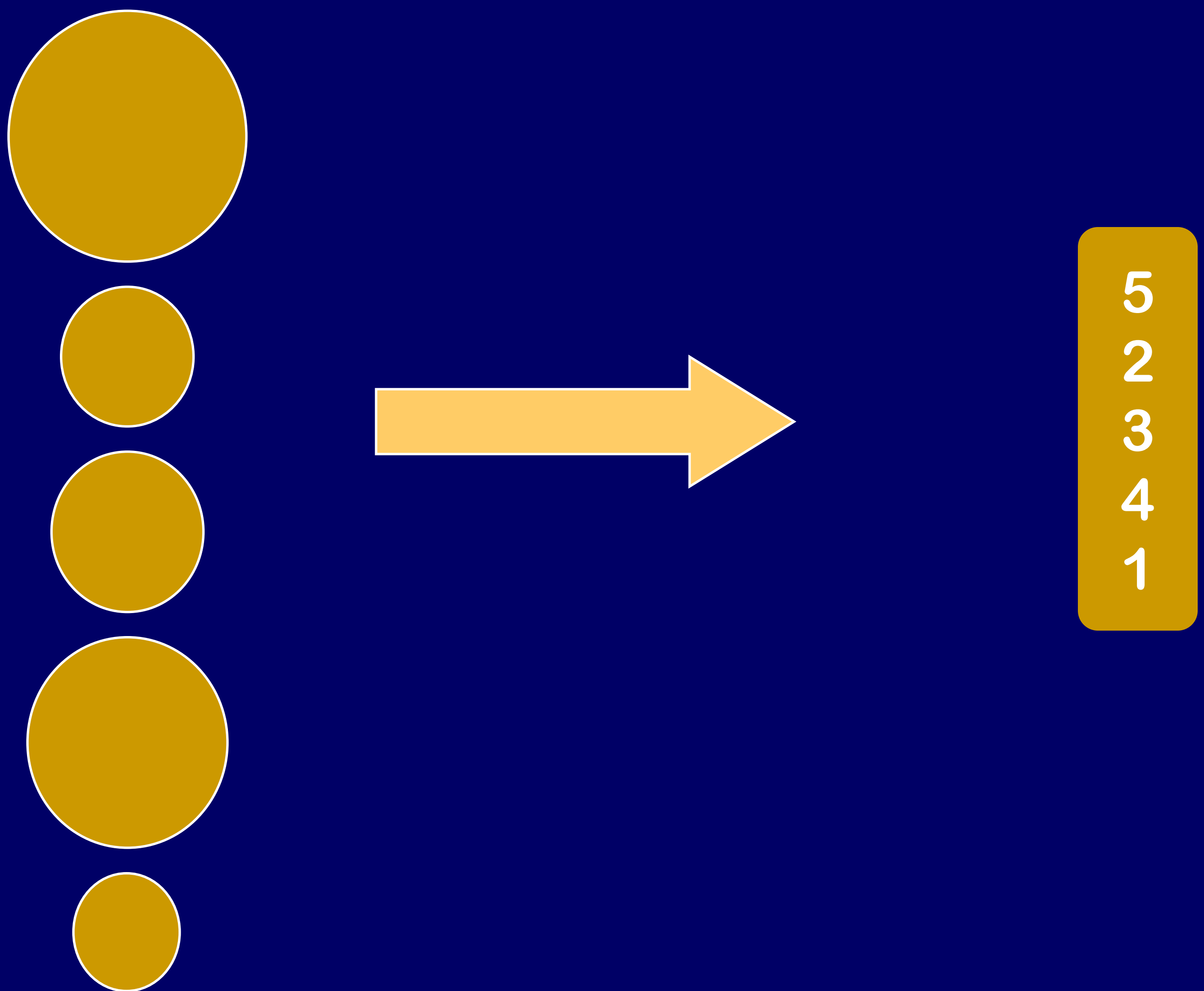
2

3

4

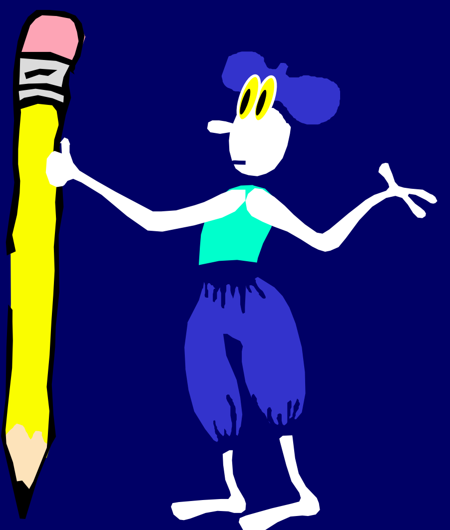
1

Developing A Notation: Turning pancakes into numbers

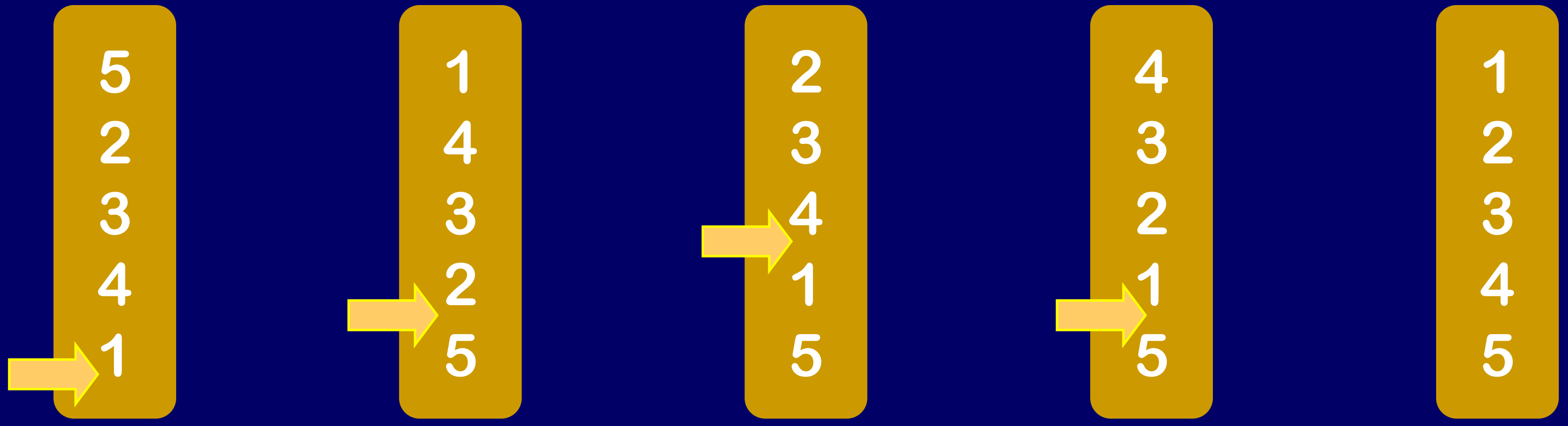


How do we sort this stack?
How many flips do we need?

5
2
3
4
1



4 Flips Are Sufficient



Algebraic Representation

X = The smallest number of flips required to sort:

5
2
3
4
1

$? \leq X \leq ?$

Upper Bound

Lower Bound

Algebraic Representation

X = The smallest number of flips required to sort:

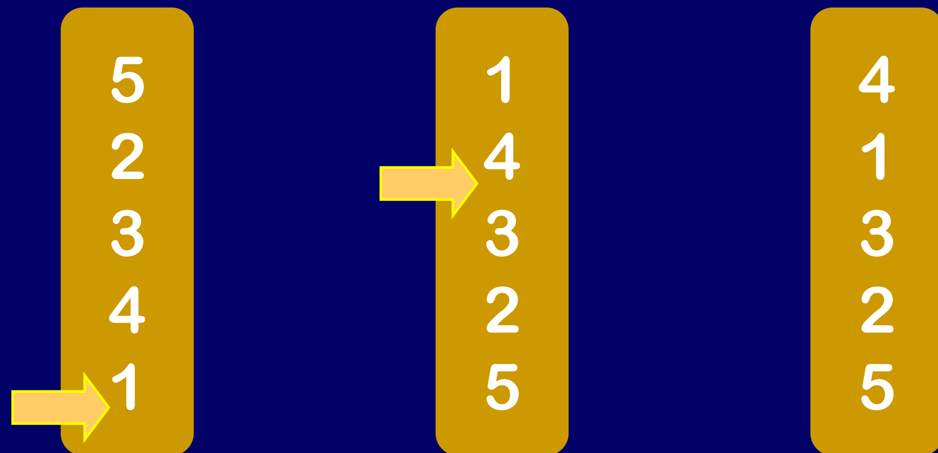
5
2
3
4
1

$? \leq X \leq 4$

Upper Bound

Lower Bound

4 Flips Are Necessary



If we could do it in 3 flips

Flip 1 has to put 5 on bottom

Flip 2 must bring 4 to top (if it didn't we'd need more than 3 flips).

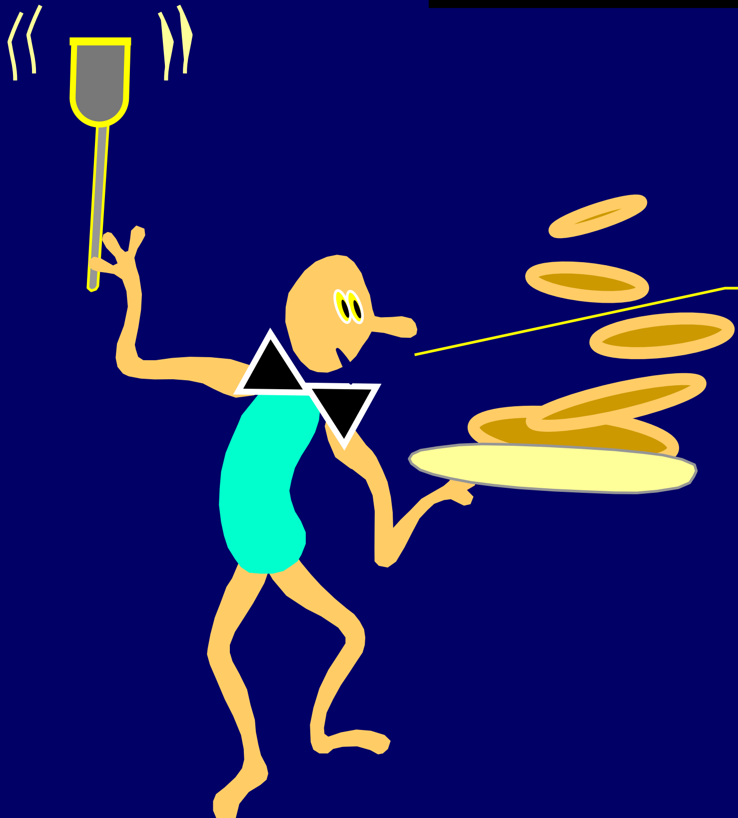
$$? \leq X \leq 4$$

Lower Bound

$$4 \leq X \leq 4$$

Lower Bound

Upper Bound



$X = 4$

5^{th} Pancake Number

$P_5 =$ The number of flips required to sort the worst case stack of 5 pancakes.

$$? \leq P_5 \leq ?$$

Upper Bound

Lower Bound

5^{th} Pancake Number

P_5 = The number of flips required to sort the worst case stack of 5 pancakes.

$$4 \leq P_5 \leq ?$$

Upper Bound

Lower Bound

5^{th} Pancake Number

P_5 = The number of flips required to sort the worst case stack of 5 pancakes.

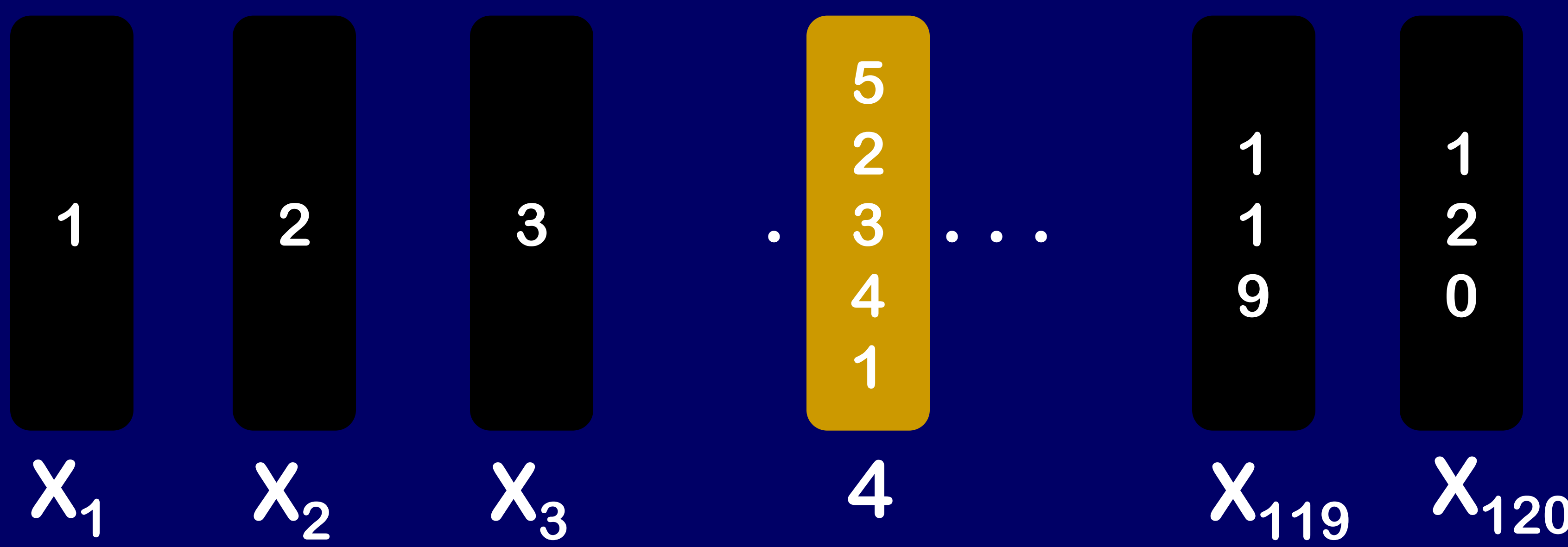
$$4 \le P_5 \le ?$$

Upper Bound

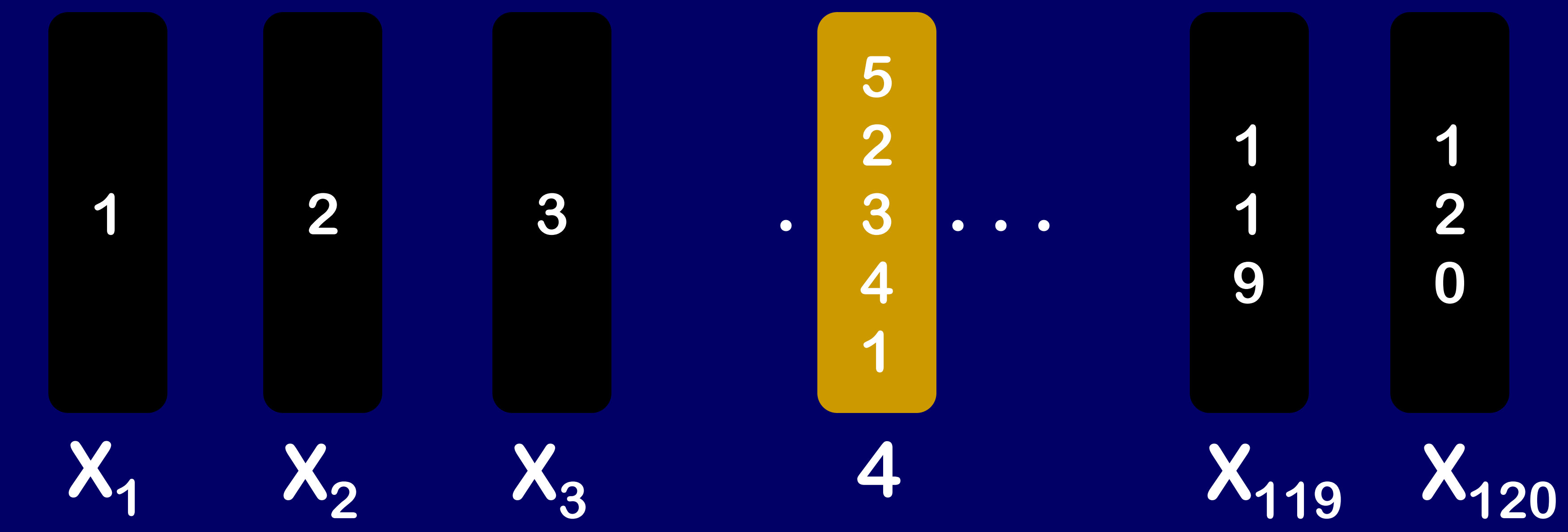
Lower Bound

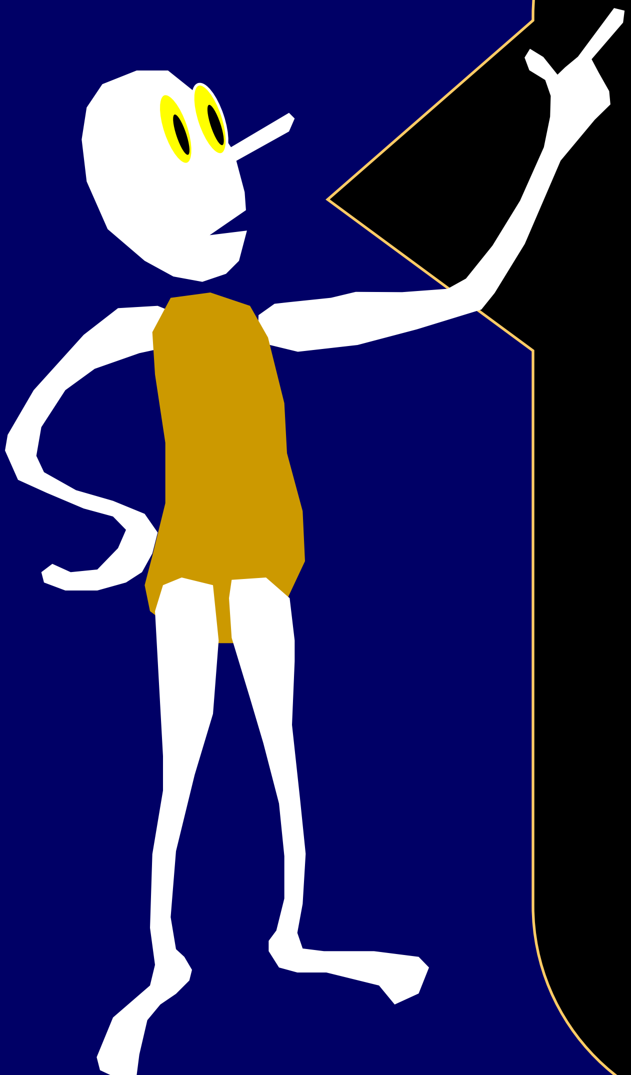
How many different pancake stacks are there with 5 unique elements?

The 5th Pancake Number: The MAX of the X's



P_5 = MAX over s stacks of 5 of MIN # of flips to sort s





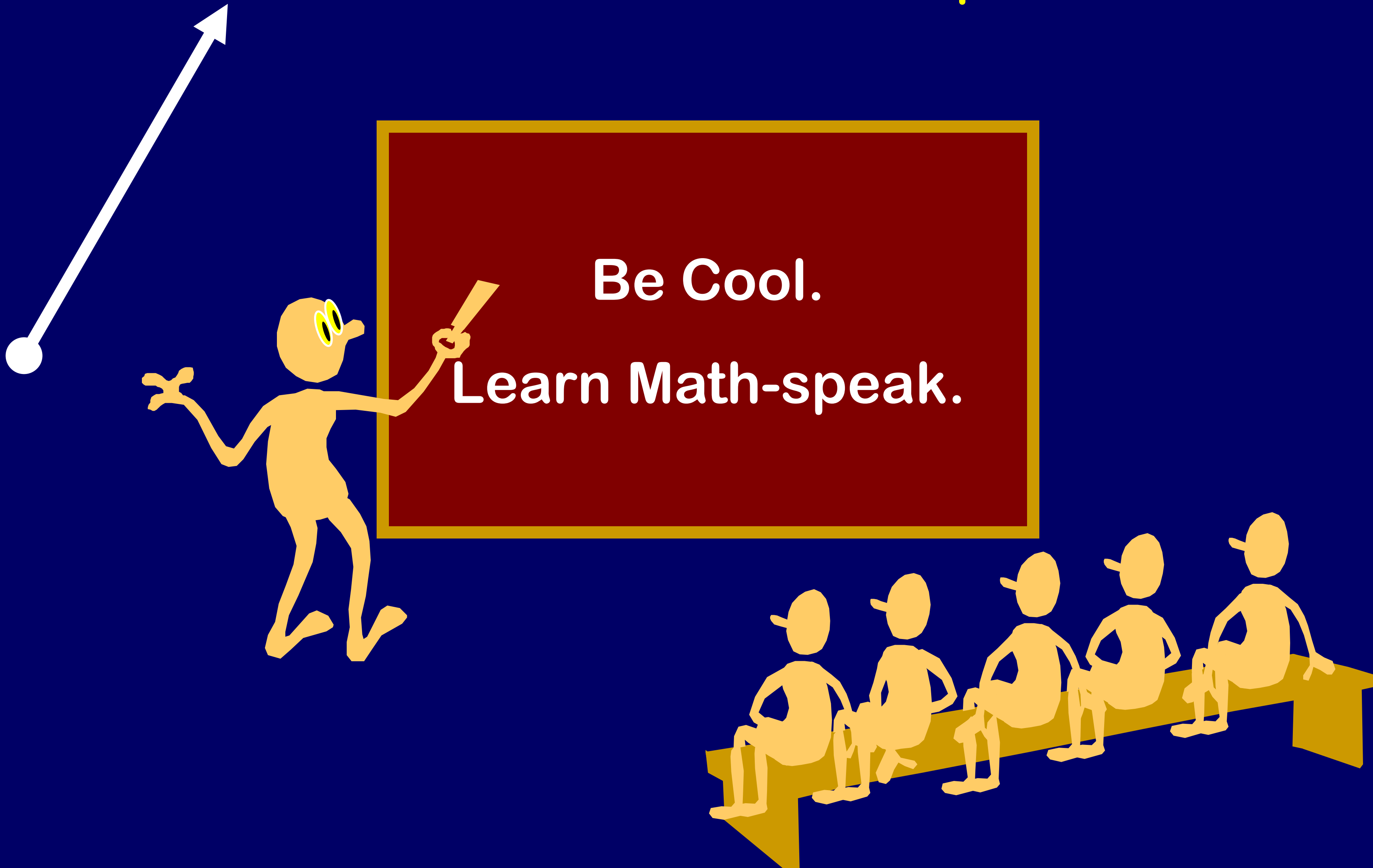
P_n

MAX over $s \in$ stacks of n pancakes of MIN # of flips to sort s

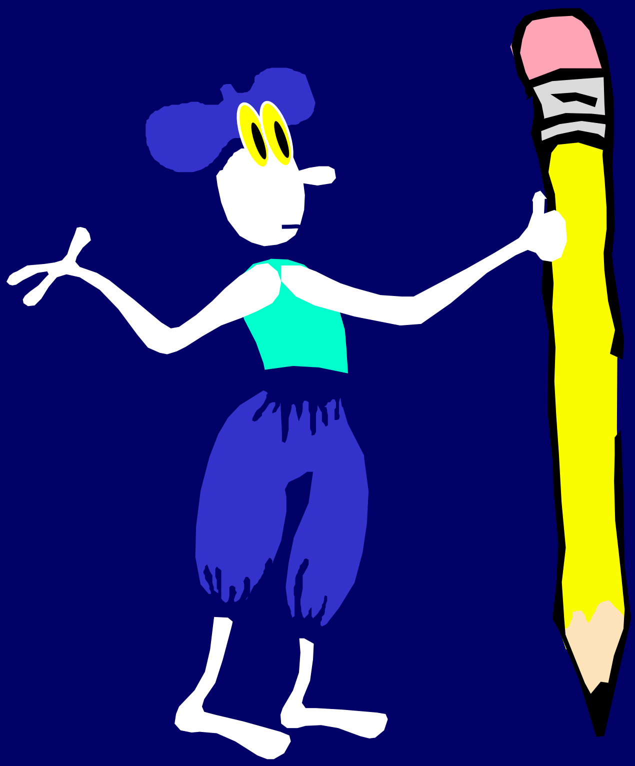
Or,

The number of flips required to sort a <u>worst-case</u> stack of n pancakes.

P_n = The number of flips required to sort <u>a</u> worst-case stack of n pancakes.



What is P_n for small n?



Can you do
n = 0, 1, 2, 3 ?

Initial Values Of P_n

n	0	1	2	3
P_n	0	0	1	3

$P_3 = 3$

1
3
2 requires 3 Flips, hence P_3 ¸ 3.

<u>ANY</u> stack of 3 can be done in 3 flips.

Get the big one to the bottom (· 2 flips).

Use · 1 more flip to handle the top two.

Hence, P_3 · 3.

n^{th} Pancake Number

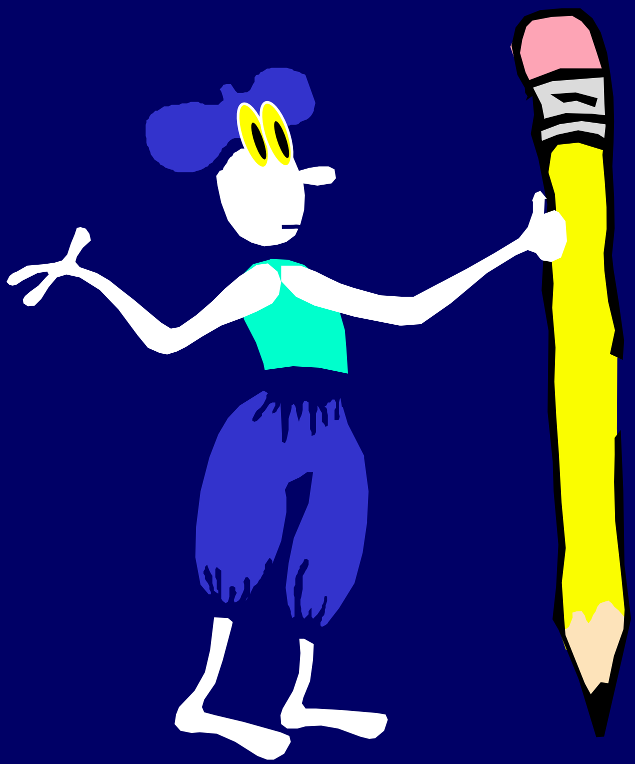
P_n = Number of flips required to sort a worst case stack of n pancakes.

$$? \leq P_n \leq ?$$

Upper Bound

Lower Bound

$$? \leq P_n \leq ?$$



Take a few minutes to try and prove bounds on P_n, for $n>3$.

Bring To Top Method

Bring biggest to top. Place it on bottom. Bring next largest to top. Place second from bottom. And so on…

Upper Bound On P_n: *Bring To Top Method For n Pancakes*

If n=1, no work - we are done.

Else: flip pancake n to top and then
flip it to position n.

Now use: Bring To Top Method For n-1 Pancakes

Total Cost: at most 2(n-1) = 2n –2 flips.

Better Upper Bound On P_n:
Bring To Top Method For n Pancakes

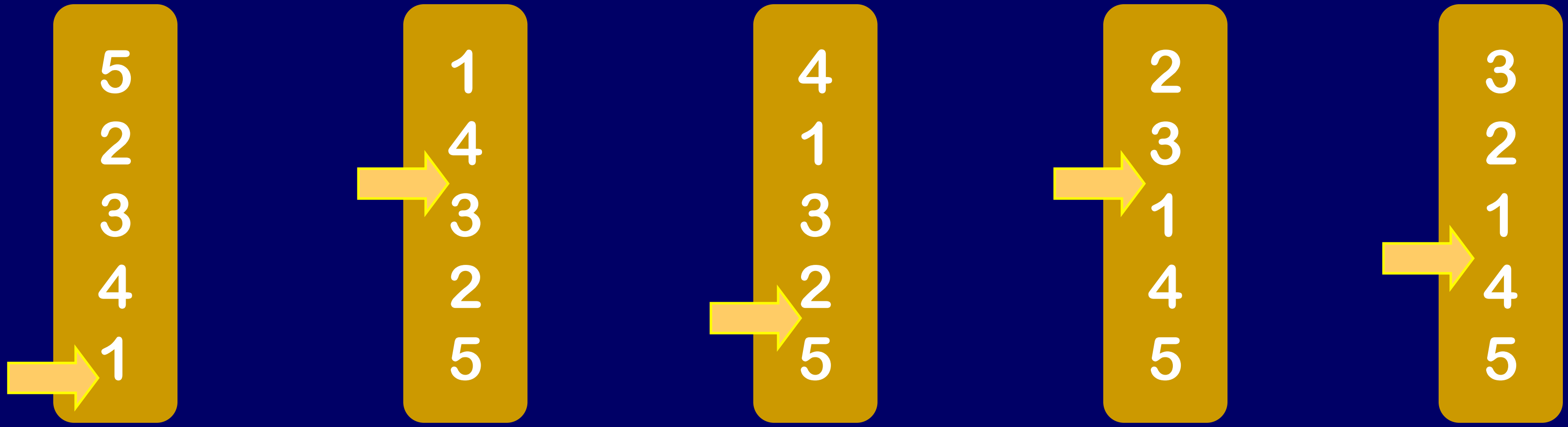
If n=2, use one flip and we are done.

Else: flip pancake n to top and then
flip it to position n.

Now use: Bring To Top Method For n-1 Pancakes

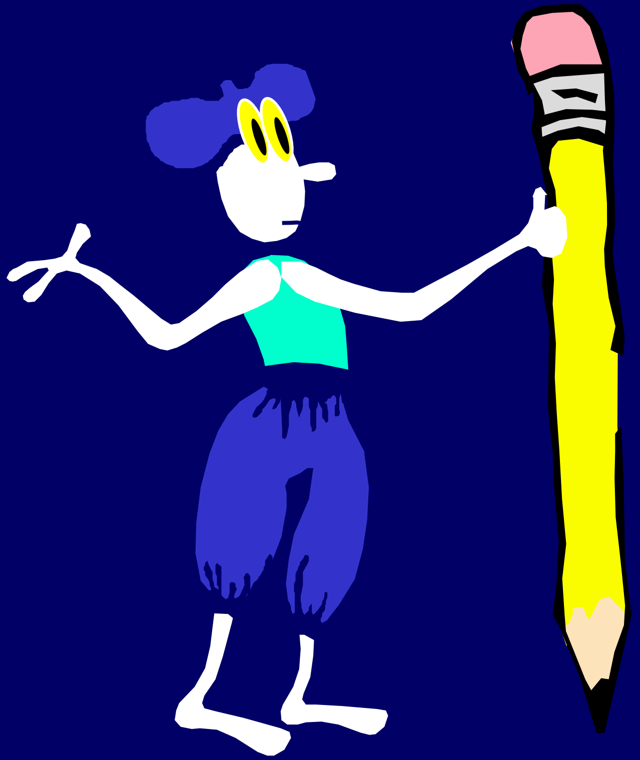
Total Cost: at most 2(n-2) + 1 = 2n –3 flips.

Bring to top not always optimal for a particular stack



5 flips, but can be done in 4 flips

$? \leq P_n \leq 2n - 3$

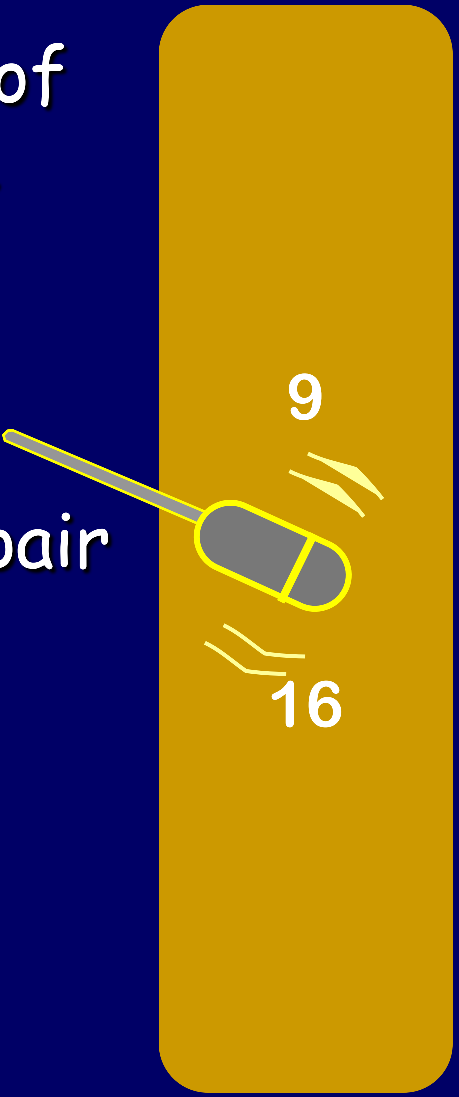


What bounds can you prove on P_n?

Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.



Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

Furthermore, this same principle is true of the "pair" formed by the bottom pancake of S and the plate.

$n \leq P_n$

Suppose n is even.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

S

2
4
6
8
.
.
n
1
3
5
7
.
.
n-1

$n \leq P_n$

Suppose n is even.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>2

$n \leq P_n$

Suppose n is odd.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

S

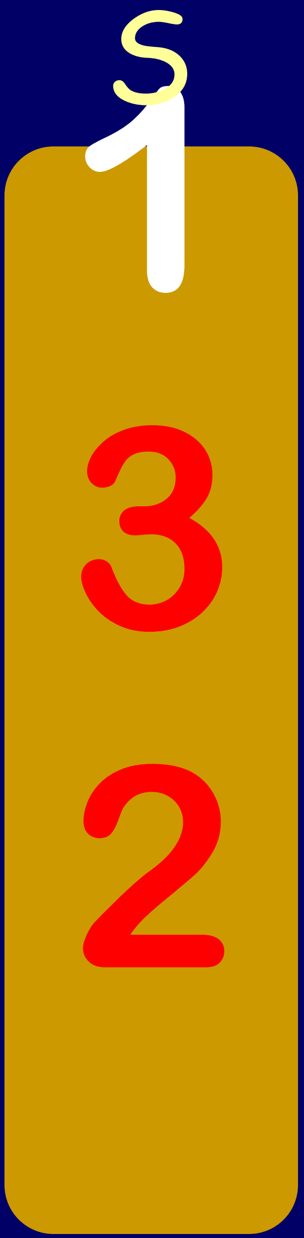
1
3
5
7
.
.
n
2
4
6
8
.
.
n-1

$n \leq P_n$

Suppose n is odd.

Such a stack S contains n pairs that must be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>3



$n \leq P_n \leq 2n - 3$ (for $n \geq 3$)

Bring To Top is within a factor of two of optimal!

$n \le P_n \le 2n - 3$ (for $n \ge 3$)

So starting from ANY stack we can get to the sorted stack using no more than P_n flips.

From ANY stack to sorted stack in · P_n.

From sorted stack to ANY stack in · P_n ?

Reverse the sequences we use to sort.

From ANY stack to sorted stack in $\cdot\ P_n$.

From sorted stack to ANY stack in $\cdot\ P_n$.

Hence,

From ANY stack to ANY stack in $\cdot\ 2P_n$.

From ANY stack to ANY stack in · $2P_n$.

Can you find a faster way than $2P_n$ flips to go from ANY to ANY?

From ANY Stack S to ANY stack T in · P_n

Rename the pancakes in T to be 1,2,3,..,n.

T : 5, 2, 4, 3, 1

T_{new} : 1, 2, 3, 4, 5

$\pi(5)$, $\pi(2)$, $\pi(4)$, $\pi(3)$, $\pi(1)$

Rewrite S using $\pi(1)$,$\pi(2)$,..,$\pi(n)$

S : 4, 3, 5, 1, 2

S_{new} : $\pi(4)$, $\pi(3)$, $\pi(5)$, $\pi(1)$, $\pi(2)$

3, 4, 1, 5, 2

From ANY Stack S to ANY stack T in · P_n

T : 5, 2, 4, 3, 1
T_{new} : 1, 2, 3, 4, 5

S : 4, 3, 5, 1, 2
S_{new} : 3, 4, 1, 5, 2

The sequence of steps that brings S_{new} to T_{new} (sorted stack) also brings S to T

The Known Pancake Numbers

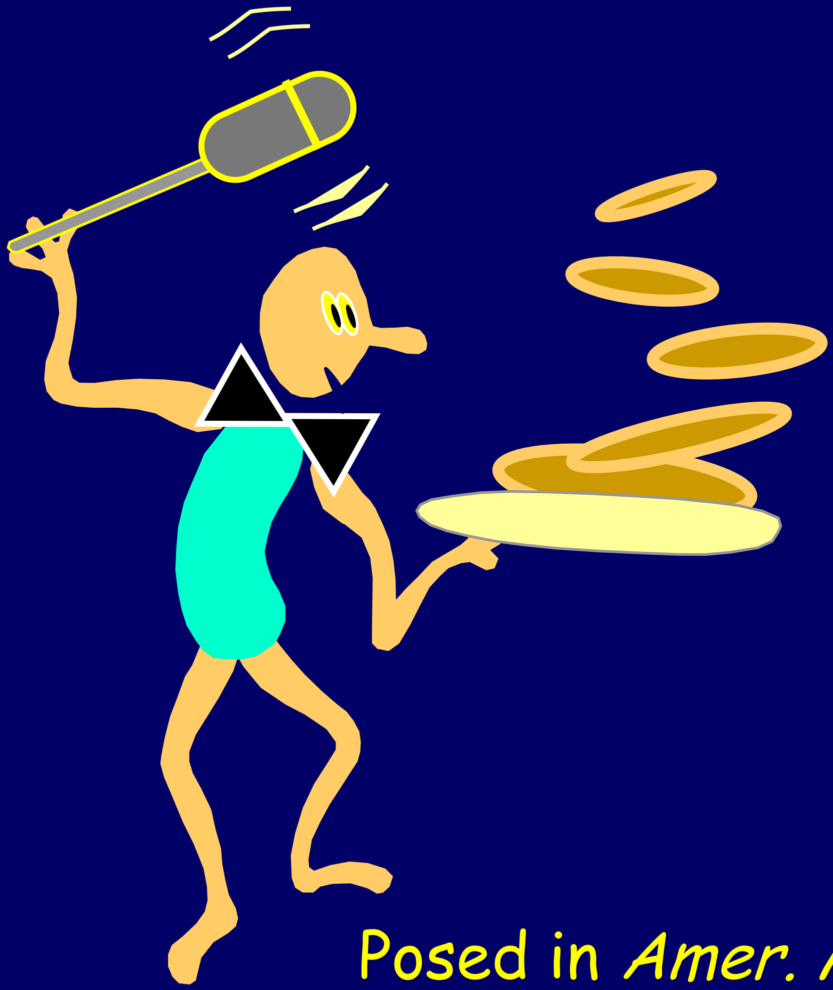
n	P_n
1	0
2	1
3	3
4	4
5	5
6	**7**
7	**8**
8	**9**
9	**10**
10	**11**
11	**13**
12	**14**
13	**15**

P_{14} Is Unknown

14! Orderings of 14 pancakes.

14! = 87,178,291,200

Is This Really Computer Science?



Posed in *Amer. Math. Monthly* 82 (1) (1975),
"Harry Dweighter" a.k.a. Jacob Goodman

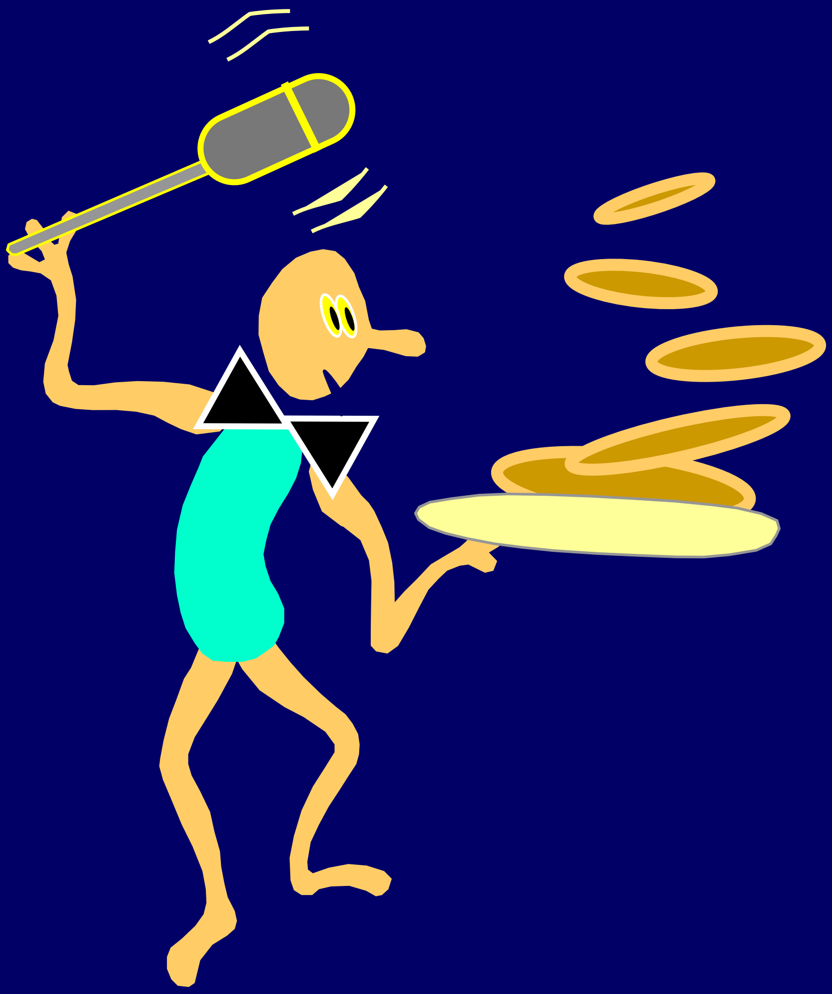
$(17/16)n \leq P_n \leq (5n+5)/3$

Bill Gates & Christos Papadimitriou:

Bounds For Sorting By Prefix Reversal.

Discrete Mathematics,
vol 27, pp 47-57, 1979.

$(15/14)n \le P_n \le (5n+5)/3$



H. Heydari & Ivan H. Sudborough.

On the Diameter of the Pancake Network.

Journal of Algorithms, vol 25, pp 67-94, 1997.

Permutation

Any particular ordering of all n elements of an n element set S is called a permutation on the set S.

Example: S = {1, 2, 3, 4, 5}

Example permutation: 5 3 2 4 1

120 possible permutations on S

Permutation

Any particular ordering of all n elements of an n element set S is called a permutation on the set S.

Each different stack of n pancakes is one of the permutations on [1..n].

Representing A Permutation

We have many choices of how to specify a permutation on S. Here are two methods:

1) List a sequence of all elements of [1..n], each one written exactly once.
   Ex: 6 4 5 2 1 3

2) Give a function π on S s.t. $\pi(1)\ \pi(2)\ \pi(3)\ ..\ \pi(n)$ is a sequence listing [1..n], each one exactly once.
   Ex: $\pi(6)=3$ $\pi(4)=2$ $\pi(5)=1$ $\pi(2)=4$ $\pi(1)=6$ $\pi(3)=5$

A Permutation is a NOUN

An ordering S of a stack of pancakes is a permutation.

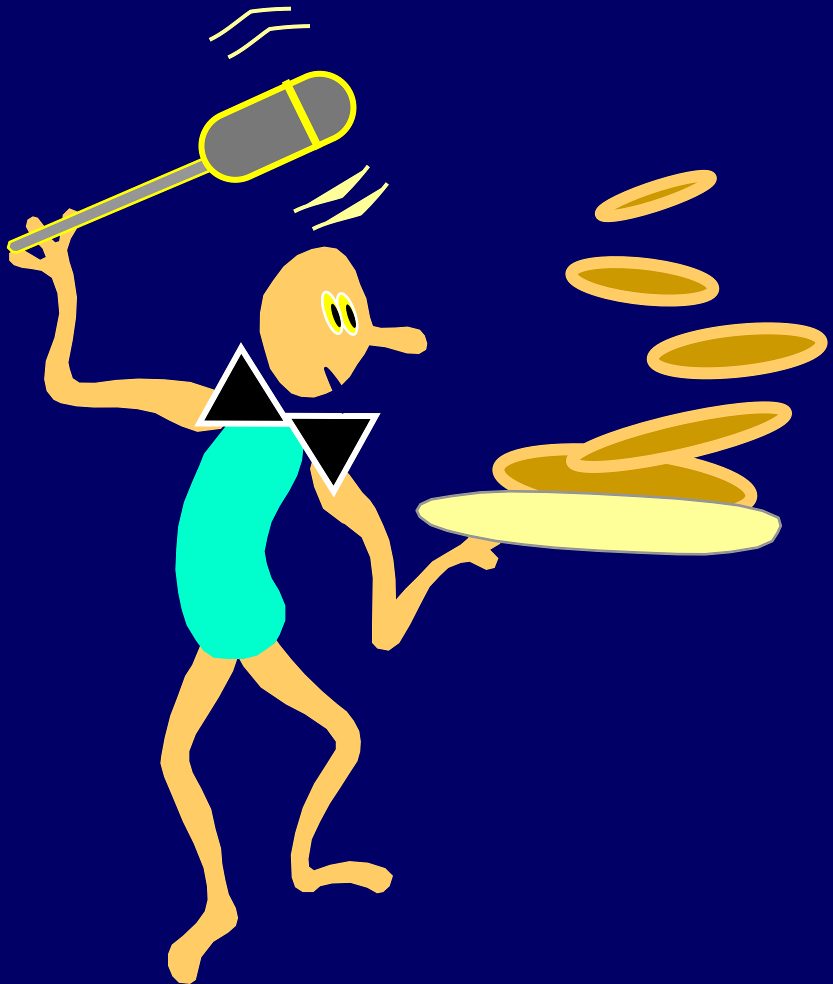
A Permutation is a NOUN
Permute is also a VERB

An ordering S of a stack of pancakes is a permutation.

We can permute S to obtain a new stack S′.

Permute also means to rearrange so as to obtain a permutation of the original.

Permute A Permutation.



I start with a permutation S of pancakes.

I continue to use a flip operation to permute my current permutation, so as to obtain the sorted permutation.

Ultra-Useful Fact

There are $n! = n*(n-1)*(n-2) \ldots 3*2*1$ permutations on n elements.

Proof: in the first counting lecture.

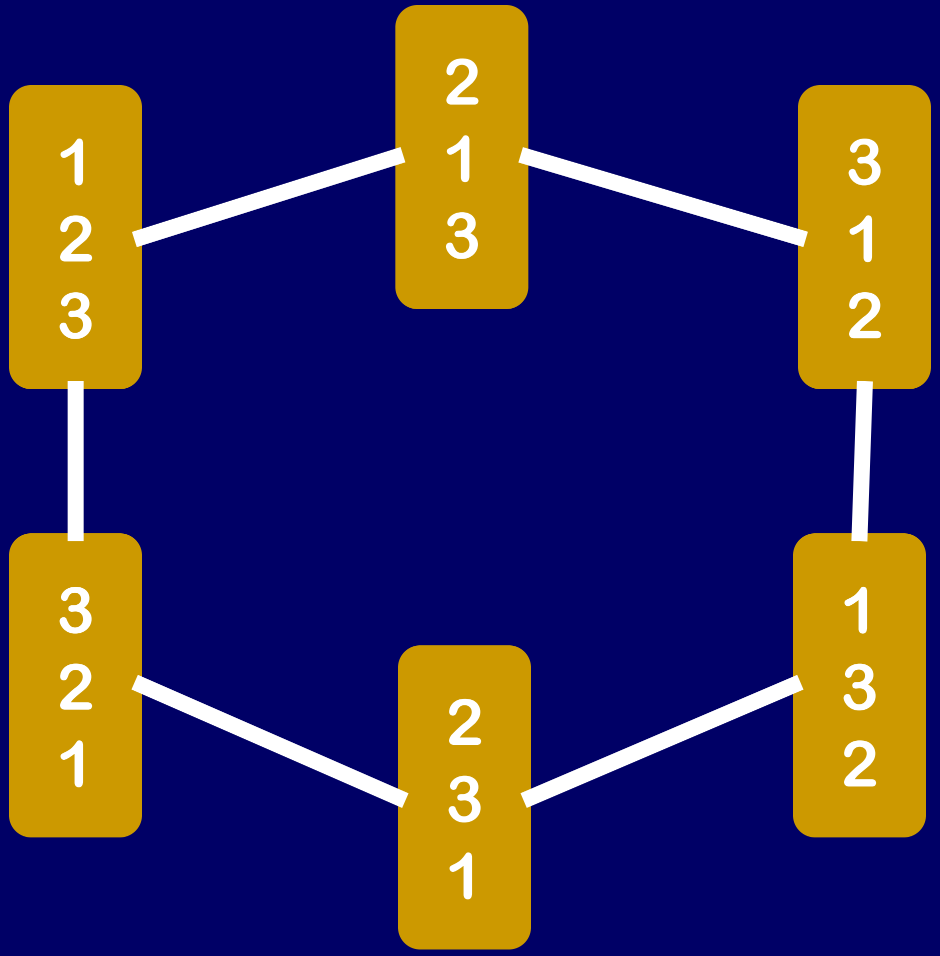
Pancake Network

This network has n! nodes

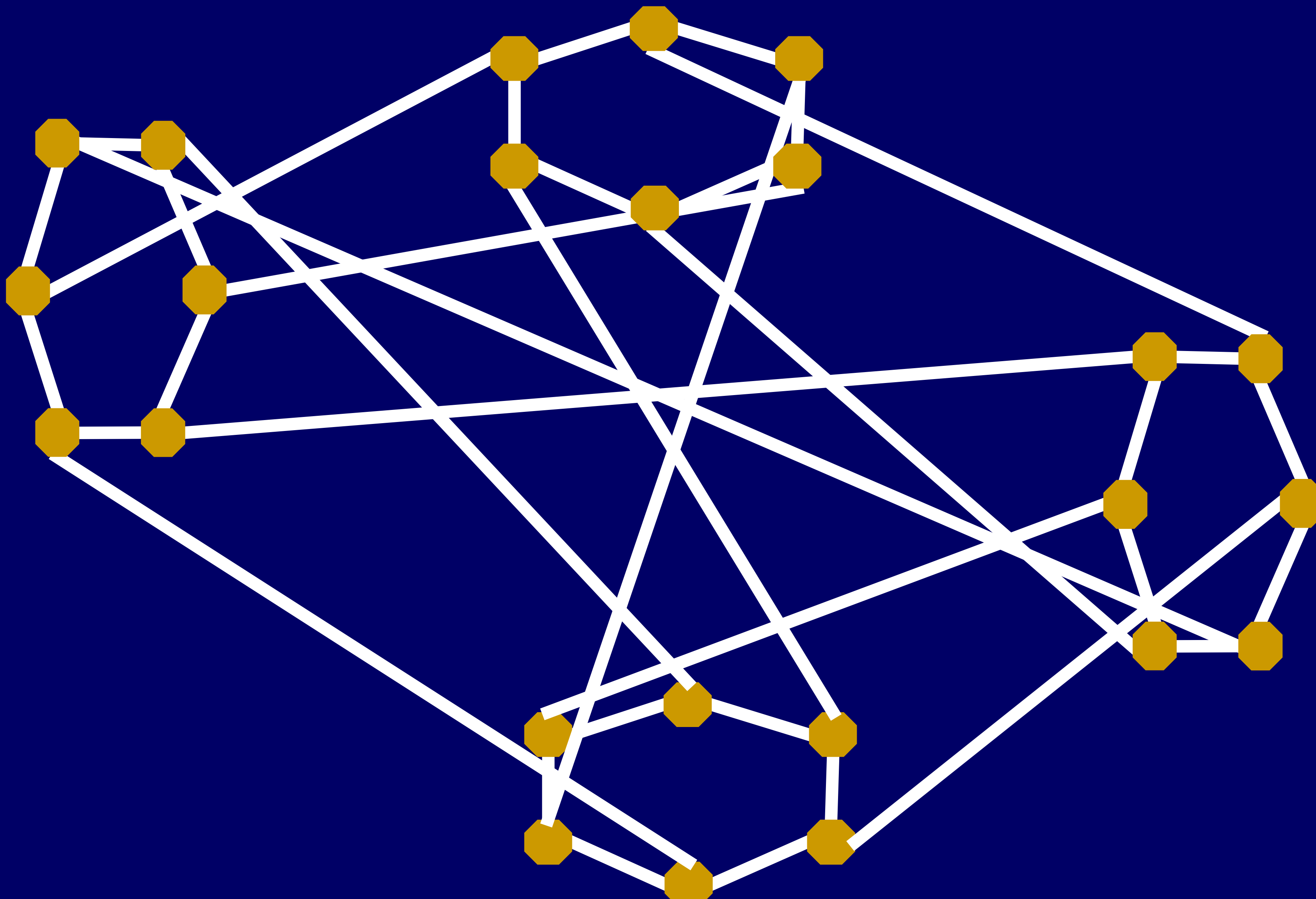
Assign each node the name of one of the possible n! stacks of pancakes.

Put a wire between two nodes
if they are one flip apart.

Network For n=3

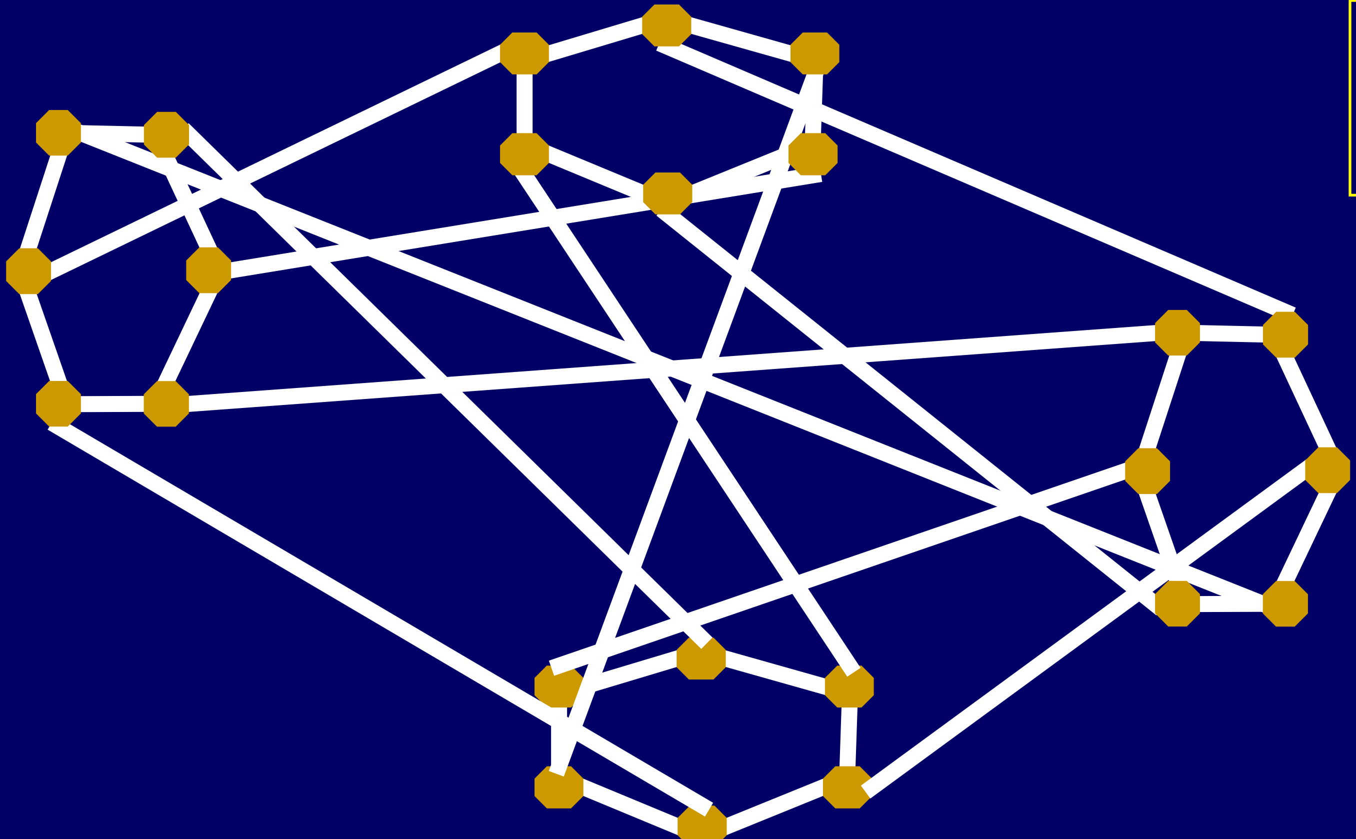


Network For n=4



Pancake Network: Routing Delay

What is the maximum distance between two nodes in the pancake network?



P_n

Pancake Network: Reliability

If up to n-2 nodes get hit by lightning the network remains connected, even though each node is connected to only n-1 other nodes.

The Pancake Network is optimally reliable for its number of edges and nodes.

Mutation Distance

Combinatorial "puzzle" to find the shortest series of reversals to transform one genome into another

Transforming Cabbage into Turnip: Polynomial Algorithm for Sorting Signed Permutations by Reversals

SRIDHAR HANNENHALLI

Bioinformatics, SmithKline Beecham Pharmaceuticals, King of Prussia, Pennsylvania

AND

PAVEL A. PEVZNER

University of Southern California, Los Angeles, California

Abstract. Genomes frequently evolve by reversals $\rho(i, j)$ that transform a gene order $\pi_1 \cdots \pi_i \pi_{i+1} \cdots \pi_{j-1} \pi_j \cdots \pi_n$ into $\pi_1 \cdots \pi_i \pi_{j-1} \cdots \pi_{i+1} \pi_j \cdots \pi_n$. Reversal distance between

Journal of the ACM, Vol. 46, No 1, 1999.

Over 350 citations!

One "Simple" Problem

A host of problems and applications at the frontiers of science

Study Bee

Definitions of:

- nth pancake number
- lower bound
- upper bound
- permutation

Proof of:

- ANY to ANY in · P_n

References

Bill Gates & Christos Papadimitriou: Bounds For Sorting By Prefix Reversal. *Discrete Mathematics,* vol 27, pp 47-57, 1979.

H. Heydari & H. I. Sudborough: On the Diameter of the Pancake Network. *Journal of Algorithms,* vol 25, pp 67-94, 1997