CompSci 102
Discrete Math for Computer Science

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

January 17, 2012

Prof. Rodger

Announcements

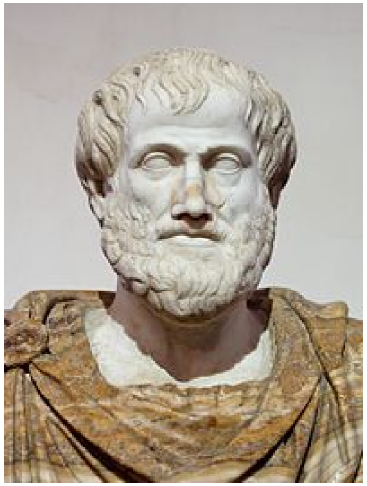
- Read for next time Chap. 1.1-1.3
- Recitations start Friday
- Added everyone to Piazza
- ACM Distinguished Speaker tonight
  - Dilma Da Silva, IBM
  - System Software for Cloud Computing
  - 6pm tonight in LSRC D106 with Pizza!

Logic

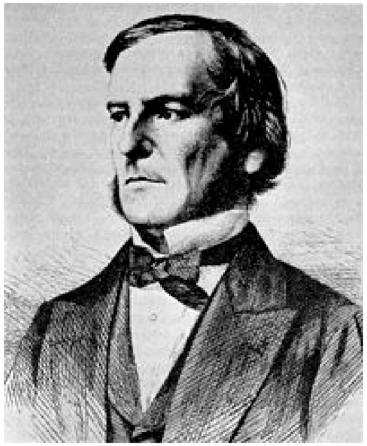
- Rules of logic specify meaning of mathematical statements
- How do you understand:
- $\forall\, n > 0 \ \sum_{i=1}^{n} i \ = \ \frac{n(n+1)}{2}$
- Applications in CS
  - Designing computers
  - Designing programming languages
  - Correctness of programs
  - Many areas such as artificial intelligence

How old …..

- Aristotle developed propositional logic over 2000 years ago….



- George Boole wrote “The Mathematical Analysis of Logic” in 1848



Proposition

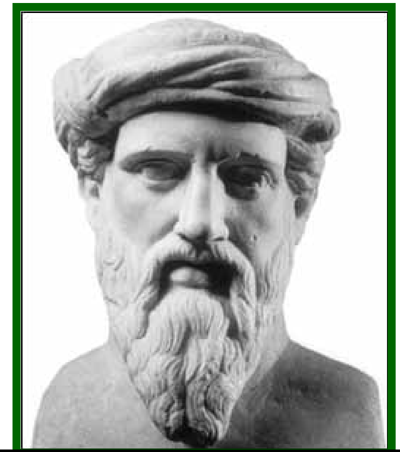
- A **proposition** is a sentence that declares a fact that is true or false

- A **theorem** is a proposition that is guaranteed by a proof

Examples of Propositions

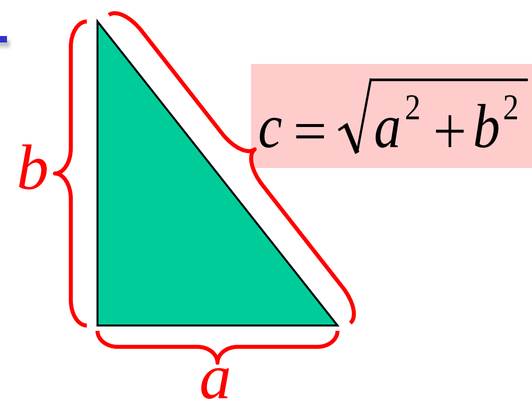
- Which are propositions? What is their value?
  1. Duke won the NCAA men's basketball title in 2010.
  2. 3x > 2
  3. Clean up after yourself.
  4. Durham is the capital of NC.
  5. Pepsi was invented in New Bern NC in 1898.
  6. 8 + 3 = 11

A Proof Example



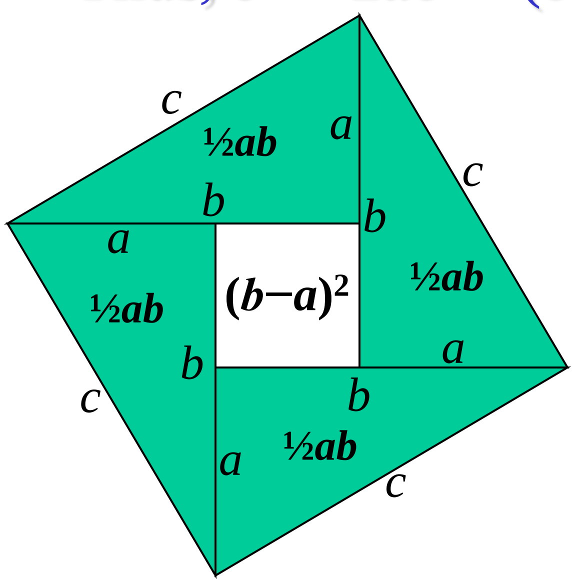
Pythagoras of Samos
(ca. 569-475 B.C.)

- **Theorem:** *(Pythagorean Theorem of Euclidean geometry)* For <u>any</u> real numbers a, b, and c, if a and b are the base-length and height of a right triangle, and c is the length of its hypo-tenuse, then $a^2 + b^2 = c^2$.
- **Proof?**



Proof of Pythagorean Theorem

- **Proof.** Consider the below diagram:
  - Exterior square area = c^2, the sum of the following regions:
    - The area of the 4 triangles = $4(\frac{1}{2}ab) = 2ab$
    - The area of the small interior square = $(b-a)^2 = b^2-2ab+a^2$.
  - Thus, $c^2 = 2ab + (b^2-2ab+a^2) = a^2 + b^2$. ■



Note: It is easy to show that the exterior and interior quadrilaterals in this construction are indeed squares, and that the side length of the internal square is indeed $b-a$ (where b is defined as the length of the longer of the two perpendicular sides of the triangle). These steps would also need to be included in a more complete proof.

Areas in this diagram are in boldface; lengths are in a normal font weight.

Operators / Connectives

An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (*E.g.*, "+" in numeric exprs.)

- *Unary* operators take 1 operand (*e.g.*, −3); *binary* operators take 2 operands (*eg* 3 × 4).
- *Propositional* or *Boolean* operators operate on propositions (or their truth values) instead of on numbers.

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

The unary *negation operator* "¬" (*NOT*) transforms a prop. into its logical *negation*.

E.g. If p = "I have brown hair."

then $\neg p$ = "I do **not** have brown hair."

The *truth table* for NOT:

p	$\neg p$
T	F
F	T

Operand column

Result column

T :≡ True; F :≡ False

":≡" means "is defined as"

The Conjunction Operator

The binary *conjunction operator* "$\wedge$" (*AND*) combines two propositions to form their logical *conjunction*.

$\wedge$ND

E.g. If p="I will have salad for lunch." and q="I will have steak for dinner.", then $p \wedge q$="I will have salad for lunch **and** I will have steak for dinner."

Remember: "$\wedge$" points up like an "A", and it means "$\wedge$ND"

Conjunction Truth Table

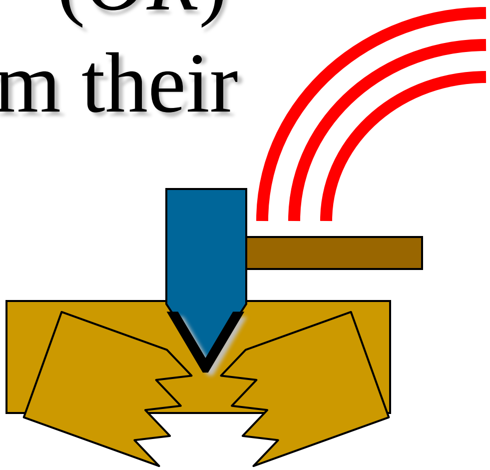
- A conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have how many rows in its truth table?
- Note: ¬ and ∧ operations together are suffi-cient to express *any* Boolean truth table!

Operand columns

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

The Disjunction Operator

The binary *disjunction operator* "$\lor$" (*OR*) combines two propositions to form their logical *disjunction*.



p="My car has a bad engine."

q="My car has a bad carburetor."

$p \lor q$="Either my car has a bad engine, **or** my car has a bad carburetor."

Meaning is like "and/or" in English.

After the downward-pointing "axe" of "$\lor$" splits the wood, you can take 1 piece OR the other, or both.

Disjunction Truth Table

- Note that $p \lor q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called *inclusive or,* because it **includes** the possibility that both p and q are true.
- "$\neg$" and "$\lor$" together are also universal.

p	q	$p \lor q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note difference from AND

Nested Propositional Expressions

- Use parentheses to *group sub-expressions*: "I just saw my old *f*riend, and either he's *g*rown or I've *s*hrunk." = $f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
  - $f \wedge g \vee s$ would be ambiguous
- By convention, "$\neg$" takes *precedence* over both "$\wedge$" and "$\vee$".
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, **not** $\neg (s \wedge f)$

A Simple Exercise

Let p=“It rained last night”,
q=“The sprinklers came on last night,”
r=“The lawn was wet this morning.”

Translate each of the following into English:

$\neg p$ = “It didn’t rain last night.”

$r \wedge \neg p$ = “The lawn was wet this morning, and it didn’t rain last night.”

$\neg r \vee p \vee q$ = “Either the lawn wasn’t wet this morning, or it rained last night, or the sprinklers came on last night.”

The *Exclusive Or* Operator

The binary *exclusive-or operator* "$\oplus$" (*XOR*) combines two propositions to form their logical "exclusive or" (exjunction?).

p = "I will earn an A in this course,"

q = "I will drop this course,"

$p \oplus q$ = "I will either earn an A in this course, or I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called *exclusive or*, because it **excludes** the possibility that both p and q are true.
- "¬" and "⊕" together are **not** universal.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note difference from OR.

Natural Language is Ambiguous

Note that <u>English</u> "or" can be <u>ambiguous</u> regarding the "both" case!

p	q	p "or" q
F	F	F
F	T	T
T	F	T
T	T	?

"Pat is a singer or Pat is a writer." - $\vee$

"Pat is a man or Pat is a woman." - $\oplus$

Need context to disambiguate the meaning!

For this class, assume "or" means <u>inclusive</u>.

The *Implication* Operator

The *implication* $\overbrace{p}^{\text{antecedent}} \rightarrow \overbrace{q}^{\text{consequent}}$ states that p implies q.

I.e., If p is true, then q is true; but if p is not true, then q could be either true or false.

E.g., let p = "You study hard."
q = "You will get a good grade."

$p \rightarrow q$ = "If you study hard, then you will get a good grade." (else, it could go either way)

Implication Truth Table

- $p \to q$ is **false** <u>only</u> when p is true but q is **not** true.
- $p \to q$ does **not** say that p <u>causes</u> q!
- $p \to q$ does **not** require that p or q **<u>are ever true</u>**!
- *E.g.* "(1=0) $\to$ pigs can fly" is TRUE!

p	q	$p \to q$
F	F	T
F	T	T
T	F	**F**
T	T	T

The <u>only</u> False case!

Examples of Implications

- "If this lecture ever ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1+1=6, then Bush is president." *True* or *False*?
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*?

Why does this seem wrong?

- Consider a sentence like,
  - "If I wear a red shirt tomorrow, then I will win the lottery!"
- In logic, we consider the sentence **True** so long as either I don't wear a red shirt, or I win the lottery.
- But, in normal English conversation, if I were to make this claim, you would think that I was lying.
  - Why this discrepancy between logic & language?

Resolving the Discrepancy

- In English, a sentence "if *p* then *q*" usually really *implicitly* means something like,
  - "In all possible situations, if *p* then *q*."
    - That is, "For *p* to be true and *q* false is *impossible*."
    - Or, "I *guarantee* that no matter what, if *p*, then *q*."
- This can be expressed in *predicate logic* as:
  - "For all situations *s*, if *p* is true in situation *s*, then *q* is also true in situation *s*"
  - Formally, we could write: $\forall s, P(s) \rightarrow Q(s)$
- *That* sentence is logically ***False*** in our example, because for me to wear a red shirt and for me to not win the lottery is a *possible* (even if not actual) situation.
  - Natural language and logic then agree with each other.

English Phrases Meaning $p \rightarrow q$

- "*p* implies *q*"
- "if *p*, then *q*"
- "if *p*, *q*"
- "when *p*, *q*"
- "whenever *p*, *q*"
- "*q* if *p*"
- "*q* when *p*"
- "*q* whenever *p*"
- "*p* only if *q*"
- "*p* is sufficient for *q*"
- "*q* is necessary for *p*"
- "*q* follows from *p*"
- "*q* is implied by *p*"

We will see some equivalent logic expressions later.

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$:

- Its *converse* is: $q \rightarrow p$.
- Its *inverse* is: $\neg p \rightarrow \neg q$.
- Its *contrapositive*: $\neg q \rightarrow \neg p$.
- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which?

How do we know for sure?

Proving the equivalence of $p \rightarrow q$ and its contrapositive using truth tables:

p	q	$\neg q$	$\neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

The *biconditional* operator

The *biconditional* $p \leftrightarrow q$ states that p is true *if and only if (IFF)* q is true.

When we say **P if and only if q** , we are saying that P *says the same thing* as Q.

Examples?

Truth table?

Biconditional Truth Table

- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- Note this truth table is the exact **opposite** of $\oplus$'s!

  Thus, $p \leftrightarrow q$ means $\neg(p \oplus q)$
- $p \leftrightarrow q$ does **not** imply that p and q are true, or that either of them causes the other, or that they have a common cause.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators. Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Some Alternative Notations

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\rightarrow$	$\leftrightarrow$
Boolean algebra:	$\overline{p}$	pq	$+$	$\oplus$		
C/C++/Java (wordwise):	!	&&	$\vert\vert$	!=		==
C/C++/Java (bitwise):	~	&	$\vert$	^		
Logic gates: