CompSci 102
Discrete Math for Computer Science

January 24, 2012

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Slides modified from Rosen

Announcements

- Read for next time Chap. 2.1-2.2
- Homework 1 due next time
- Bala is the TA leading recitations until Feb 6
- UTA Jimmy Mu will have one night in the link
- Finish notes from last time ….

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is
  $$\forall x \exists y(x + y = 0)$$
  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:
  $\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$
  is $\quad \exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.

      $\forall x\, \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.

      $\forall x \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ do they have the same truth value?

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then do $\forall x \exists y P(x,y)$ and $\exists y\, \forall x P(x,y)$ have the same truth value?.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ do they have the same truth value?

   Both True

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then do $\forall x \exists y P(x,y)$ and $\exists y\, \forall x P(x,y)$ have the same truth value?.

   First True, Second one False

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:
1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y\, P(x,y)$

4. $\exists x \exists y\, P(x,y)$

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:
1. $\forall x \forall y P(x,y)$
   Answer: False

2. $\forall x \exists y P(x,y)$
   Answer: True

3. $\exists x \forall y\, P(x,y)$
   Answer: True

4. $\exists x \exists y\, P(x,y)$
   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,
Define $P(x,y) : x / y = 1$
What is the truth value of the following:
1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y P(x,y)$

4. $\exists x \exists y P(x,y)$

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,
Define $P(x,y) : x / y = 1$
What is the truth value of the following:
1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: False, x = 0, 0/y = 0
3. $\exists x \forall y P(x,y)$
   Answer: False
4. $\exists x \exists y P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y

Translating Nested Quantifiers into English

Example 1: Translate the statement
$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$
where C(x) is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Example 1: Translate the statement
$$\exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Translating Nested Quantifiers into English

Example 1: Translate the statement

$\forall x \ (C(x)\lor \exists y \ (C(y) \land F(x, y)))$

where C(x) is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 1: Translate the statement

$\exists x \ \forall y \ \forall z \ ((F(x, y)\land F(x,z) \land (y \neq z))\rightarrow \neg F(y,z))$

Solution: Every student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

2. Introduce the variables x and y, and specify the domain, to obtain:

3. The result is:

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables x and y, and specify the domain, to obtain:
   "For all positive integers x and y, $x + y$ is positive."
3. The result is:
   $\forall x \ \forall \ y ((x > 0)\land (y > 0)\rightarrow (x + y > 0))$
   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

Translating English into Logical Expressions Example

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2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \,\forall a \,\exists f \,(P(w,f) \land Q(f,a))$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Example 2: "Siblinghood is symmetric."

Example 3: "Everybody loves somebody."

Example 4: "There is someone who is loved by everyone."

Example 5: "There is someone who loves someone."

Example 6: "Everyone loves himself"

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."
  Solution: $\forall x \,\forall y \,(B(x,y) \to S(x,y))$

Example 2: "Siblinghood is symmetric."
  Solution: $\forall x \,\forall y \,(S(x,y) \to S(y,x))$

Example 3: "Everybody loves somebody."
  Solution: $\forall x \,\exists y \,L(x,y)$

Example 4: "There is someone who is loved by everyone."
  Solution: $\exists y \,\forall x \,L(x,y)$

Example 5: "There is someone who loves someone."
  Solution: $\exists x \,\exists y \,L(x,y)$

Example 6: "Everyone loves himself"
  Solution: $\forall x \,L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:
$$\exists w \,\forall a \,\exists f \,(P(w,f) \land Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.
  :

Part 3: Can you translate the result back into English?

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:
$$\exists w \,\forall a\, \exists f\ (P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \,\forall a\, \exists f\ (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:
1. $\neg \exists w \,\forall a\, \exists f\ (P(w,f) \wedge Q(f,a))$
2. $\forall w \,\neg\, \forall a\, \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \,\exists a\, \neg\, \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \,\exists a\, \forall f\, \neg\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \,\exists a\, \forall f\ (\neg\, P(w,f) \vee \neg\, Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Some Questions about Quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

Some Questions about Quantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$
    Solution: Yes The order in which x and y are picked does not matter.

  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$
    Solution: No! Try "x + y = 0" for $P(x,y)$ with U being the integers. The order in which the values of x and y are picked does matter.

- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
    Solution: Yes!

  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$
    Solution: No. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

Revisiting the Socrates Example

- We have the two premises:
  - "All men are mortal."
  - "Socrates is a man."
- And the conclusion:
  - "Socrates is mortal."
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\forall x(Man(x) \rightarrow Mortal(x))$$
$$Man(Socrates)$$
$$\therefore \quad Mortal(Socrates)$$

- We will see shortly that this is a valid argument.

Valid Arguments

- We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

1. Propositional Logic

   Inference Rules

2. Predicate Logic

   Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion. An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p_1, p_2, ..., p_n$ and the conclusion is q then

  $(p_1 \wedge p_2 \wedge ... \wedge p_n) \rightarrow q$ is a tautology.
- Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic: Modus Ponens

$$p \rightarrow q$$
$$p$$
$$\therefore q$$

Corresponding Tautology:
$(p \wedge (p \rightarrow q)) \rightarrow q$

Example:
Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore, I will study discrete math."

Modus Tollens

$$p \to q$$
$$\neg q$$
$$\overline{\therefore \neg p}$$

Corresponding Tautology:
$$(\neg q \wedge (p \to q)) \to \neg p$$

Example:
Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore, it is not snowing."

Hypothetical Syllogism

$$p \to q$$
$$q \to r$$
$$\overline{\therefore p \to r}$$

Corresponding Tautology:
$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore, If it snows, I will get an A."

Disjunctive Syllogism

$$p \vee q$$
$$\neg p$$
$$\overline{\therefore q}$$

Corresponding Tautology:
$$(\neg p \wedge (p \vee q)) \to q$$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore, I will study English literature."

Addition

$$p$$
$$\overline{\therefore p \vee q}$$

Corresponding Tautology:
$$p \to (p \vee q)$$

Example:
Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Corresponding Tautology:
$(p \wedge q) \to p$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$$

Corresponding Tautology:
$((p) \wedge (q)) \to (p \wedge q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

$$\frac{\begin{array}{c} \neg p \vee r \\ p \vee q \end{array}}{\therefore q \vee r}$$

Resolution plays an important role in AI and is used in Prolog.

Corresponding Tautology:
$((\neg p \vee r) \wedge (p \vee q)) \to (q \vee r)$

Example:
Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$S_1$$
$$S_2$$
$$.$$
$$.$$
$$.$$
$$S_n$$

$$\therefore C$$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \to q)$$

Show that q is a conclusion.

Solution:

Step	Reason
1. $p \wedge (p \to q)$	premise
2. p	Simplification using 1)
3. $(p \to q)$	Simplification using 1)
4. q	Modus ponens using 2)

Valid Arguments

Example 2:

- With these hypotheses:
    "It is not sunny this afternoon and it is colder than yesterday."
    "We will go swimming only if it is sunny."
    "If we do not go swimming, then we will take a canoe trip."
    "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the conclusion:
    "We will be home by sunset."

Solution:

1. Choose propositional variables:
    p : "It is sunny this afternoon." r : "We will go swimming." t : "We will be home by sunset."
    q : "It is colder than yesterday." s : "We will take a canoe trip."

2. Translation into propositional logic:

    Hypotheses: $\neg p \wedge q, r \to p, \neg r \to s, s \to t$
    Conclusion: t

    Continued on next slide
    →

Valid Arguments

3. Construct the Valid Argument

1. $\neg p \wedge q$	premise

Valid Arguments

3. Construct the Valid Argument

1. $\neg p \wedge q$	premise
2. $\neg p$	Simplification using 1)
3. $r \to p$	premise
4. $\neg r$	Modus tollens using 2) 3)
5. $\neg r \to s$	premise
6. s	Modus ponens with 4) 5)
7. $s \to t$	premise
8. t	Modus ponens with 6) 7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all cats and Charlotte is a cat.

"All cats are cuddly."

"Therefore, Charlotte is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her a and say that a got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	Reason
1. $\forall x(M(x) \to L(x))$	Premise
2. $M(J) \to L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."
"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote " x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\frac{\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \to P(x)) \end{array}}{\therefore \exists x(P(x) \land \neg B(x))}$$

Continued on next slide
→

Using Rules of Inference

Valid Argument:

Using Rules of Inference

Valid Argument:

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$
_____

$$\therefore \quad Mortal(Socrates)$$

Solution for Socrates Example

Valid Argument

Step	Reason
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\forall x(P(x) \rightarrow Q(x))$$
$$P(a), \text{where } a \text{ is a particular}$$
$$\underline{\text{element in the domain}}$$
$$\therefore Q(a)$$

This rule could be used in the Socrates example.