CompSci 102
Discrete Math for Computer Science

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
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February 23, 2012

Prof. Rodger

Announcements

- No Recitation tomorrow or next Friday
- Recitations start back after spring break

Chap 4.3 - Primes

Definition: A positive integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*. A positive integer that is greater than 1 and is not prime is called *composite*.

Example: The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

The Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Examples:

- 105 =
- 641 =
- 221 =
- 1024 =

The Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Examples:

- $105 = \quad 3 \cdot 5 \cdot 7$
- $641 = \quad 641$
- $221 = \quad 13 \cdot 17$
- $1024 = \quad 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$

The Sieve of Erastosthenes

Erastothenes
(276-194 B.C.)

- The *Sieve of Erastosthenes* can be used to find all primes not exceeding a specified positive integer. For example, begin with the list of integers between 1 and 100.
  a. Delete all the integers, other than 2, divisible by 2.
  b. Delete all the integers, other than 3, divisible by 3.
  c. Next, delete all the integers, other than 5, divisible by 5.
  d. Next, delete all the integers, other than 7, divisible by 7.
  e. Since all the remaining integers are not divisible by any of the previous integers, other than 1, the primes are:

continued →

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Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

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The Sieve of Erastosthenes

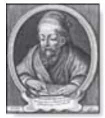
If an integer n is a composite integer, then it has a prime divisor less than or equal to $\sqrt{n}$.

To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Trial division, a very inefficient method of determining if a number n is prime, is to try every integer $i \leq \sqrt{n}$ and see if n is divisible by i.

In previous example, why did we use only 2, 3, 5 and 7?

Infinitude of Primes



Euclid
(325 B.C.E. – 265 B.C.E.)

Theorem: There are infinitely many primes. (Euclid)

Proof: Assume finitely many primes: $p_1, p_2, \ldots, p_n$

- Let $q = p_1 p_2 \cdots p_n + 1$
- Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
  - But none of the primes p_j divides q since if $p_j \mid q$, then p_j divides $q - p_1 p_2 \cdots p_n = 1$.
  - Hence, there is a prime not on the list $p_1, p_2, \ldots, p_n$. It is either q, or if q is composite, it is a prime factor of q. This contradicts the assumption that $p_1, p_2, \ldots, p_n$ are all the primes.
- Consequently, there are infinitely many primes.

Paul Erdős
(1913-1996)

This proof was given by Euclid *The Elements*. The proof is considered to be one of the most beautiful in all mathematics. It is the first proof in *The Book*, inspired by the famous mathematician Paul Erdős' imagined collection of perfect proofs maintained by God.

Mersenne Primes

Marin Mersenne
(1588-1648)

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called *Mersenne primes*.

- $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, and $2^7 - 1 = 127$ are Mersenne primes.
- $2^{11} - 1 = 2047$ is not a Mersenne prime since $2047 = 23 \cdot 89$.
- There is an efficient test for determining if $2^p - 1$ is prime.
- The largest known prime numbers are Mersenne primes.
- As of mid 2011, 47 Mersenne primes were known, the largest is $2^{43,112,609} - 1$, which has nearly 13 million decimal digits.
- The *Great Internet Mersenne Prime Search* (*GIMPS*) is a distributed computing project to search for new Mersenne Primes.

http://www.mersenne.org/

Distribution of Primes

- Mathematicians have been interested in the distribution of prime numbers among the positive integers. In the nineteenth century, the *prime number theorem* was proved which gives an asymptotic estimate for the number of primes not exceeding x.

  Prime Number Theorem: The ratio of the number of primes not exceeding x and $x/\ln x$ approaches 1 as x grows without bound. ($\ln x$ is the natural logarithm of x)
  - The theorem tells us that the number of primes not exceeding x, can be approximated by $x/\ln x$.
  - The odds that a randomly selected positive integer less than n is prime are approximately $(n/\ln n)/n = 1/\ln n$.

Generating Primes

- Finding large primes with hundreds of digits is important in cryptography.
- There is no simple function $f(n)$ such that $f(n)$ is prime for all positive integers n.
- Consider
  - $f(n) = n^2 - n + 41$ is prime for all integers 1,2,..., 40.
  - But $f(41) = 41^2$ is not prime.
- Fortunately, we can generate large integers which are almost certainly primes. See Chapter 7.

Conjectures about Primes

Many conjectures about them are unresolved, including:

- *Goldbach's Conjecture*: Every even integer n, $n > 2$, is the sum of two primes. It has been verified by computer for all positive even integers up to $1.6 \cdot 10^{18}$. The conjecture is believed to be true by most mathematicians.
- There are infinitely many primes of the form $n^2 + 1$, where n is a positive integer. But it has been shown that there are infinitely many primes of the form $n^2 + 1$, where n is a positive integer or the product of at most two primes.
- *The Twin Prime Conjecture*: The twin prime conjecture is that there are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers $65{,}516{,}468{,}355 \cdot 2^{333{,}333} \pm 1$, which have 100,355 decimal digits.

Greatest Common Divisor

Definition: Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and also $d \mid b$ is called the greatest common divisor of a and b. The greatest common divisor of a and b is denoted by $\gcd(a,b)$.

One can find greatest common divisors of small numbers by inspection.

Example: What is the greatest common divisor of 24 and 36?

Example: What is the greatest common divisor of 17 and 22?

Greatest Common Divisor

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One can find greatest common divisors of small numbers by inspection.

Example: What is the greatest common divisor of 24 and 36?

Solution: $\gcd(24,36) = 12$

Example: What is the greatest common divisor of 17 and 22?

Solution: $\gcd(17,22) = 1$

Greatest Common Divisor

Definition: The integers a and b are *relatively prime* if their greatest common divisor is 1.

Example: 17 and 22

Definition: The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \le i < j \le n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Solution:

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution:

Greatest Common Divisor

Definition: The integers a and b are *relatively prime* if their greatest common divisor is 1.

Example: 17 and 22

$\gcd(17, 22) = 1$

Definition: The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \le i < j \le n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Solution: Because $\gcd(10,17) = 1$, $\gcd(10,21) = 1$, and $\gcd(17,21) = 1$, 10, 17, and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because $\gcd(10,24) = 2$, 10, 19, and 24 are not pairwise relatively prime.

Finding the Greatest Common Divisor Using Prime Factorizations

- Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, \qquad b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots p_n^{\min(a_n,b_n)}.$$

- This formula is valid since the integer on the right (of the equals sign) divides both a and b. No larger integer can divide both a and b.
-

Example: $120 = 2^3 \cdot 3 \cdot 5$ $\quad 500 = 2^2 \cdot 5^3$

$\gcd(120,500) =$

- Finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Finding the Greatest Common Divisor Using Prime Factorizations

- Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, \qquad b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots p_n^{\min(a_n,b_n)}.$$

- This formula is valid since the integer on the right (of the equals sign) divides both a and b. No larger integer can divide both a and b.
-

Example: $120 = 2^3 \cdot 3 \cdot 5$ $\quad 500 = 2^2 \cdot 5^3$

$\gcd(120,500) = 2^{\min(3,2)} \cdot 3^{\min(1,0)} \cdot 5^{\min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$

- Finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Least Common Multiple

Definition: The least common multiple of the positive integers a and b is the smallest positive integer that is divisible by both a and b. It is denoted by $\text{lcm}(a,b)$.

- The least common multiple can also be computed from the prime factorizations.

$$\text{lcm}(a, b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}$$

This number is divided by both a and b and no smaller number is divided by a and b.

Least Common Multiple

Example: lcm(30, 35) =

Example: $\text{lcm}(2^3 3^5 7^2, 2^4 3^3)$ =

Least Common Multiple

Example: lcm(30, 35) =

5*2*3, 7*5

5*2*3*7 = 210

Example: $\text{lcm}(2^3 3^5 7^2, 2^4 3^3)$ =

$$2^{\max(3,4)}\, 3^{\max(5,3)}\, 7^{\max(2,0)} = 2^4\, 3^5\, 7^2$$

LCM and GCD relation

Theorem 5: Let a and b be positive integers. Then

$ab = \gcd(a,b) \cdot \text{lcm}(a,b)$

Example: gcd (20,15) · lcm(20,15)

Proof:

LCM and GCD relation

Theorem 5: Let a and b be positive integers. Then

$ab = \gcd(a,b) \cdot \text{lcm}(a,b)$

Example: gcd (20,15) · lcm(20,15)

$= (5^1 \cdot 2^0) \cdot (5^1 3^1 2^2) = (5) \cdot (60)$

$= 300$

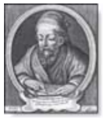
$= 20 \cdot 15$

Proof:

Note that $\min(x,y) + \max(x,y) = x + y$

one uses the larger exponent and the other one the smaller exponent, but you get all factors back.

Euclidean Algorithm



Euclid
(325 B.C.E. – 265 B.C.E.)

- The Euclidian algorithm is an efficient method for computing the greatest common divisor of two integers. It is based on the idea that gcd(*a*,*b*) is equal to gcd(*a*,*c*) when $a > b$ and *c* is the remainder when a is divided by *b*.

Example: Find gcd(91, 287):

- $287 = 91 \cdot 3 + 14$ — Divide 287 by 91
- $91 = 14 \cdot 6 + 7$ — Divide 91 by 14
- $14 = 7 \cdot 2 + 0$ — Divide 14 by 7 (Stopping condition)

gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = 7

continued →

Euclidean Algorithm

- The Euclidean algorithm expressed in pseudocode is:

```
procedure gcd(a, b: positive integers)
x := a
y := b
while  y ≠ 0
    r := x mod y
    x := y
    y := r
return x {gcd(a,b) is x}
```

Correctness of Euclidean Algorithm

Lemma 1: Let *a* = *bq* + *r*, where *a*, *b*, *q*, and *r* are integers. Then gcd(*a*,*b*) = gcd(*b*,*r*).

Proof:

– Suppose that *d* divides both *a* and *b*.

– Suppose that *d* divides both *b* and *r*.

– Therefore, gcd(*a*,*b*) = gcd(*b*,*r*).

◀

Correctness of Euclidean Algorithm

Lemma 1: Let *a* = *bq* + *r*, where *a*, *b*, *q*, and *r* are integers. Then gcd(*a*,*b*) = gcd(*b*,*r*).

Proof:

– Suppose that *d* divides both *a* and *b*.

Then *d* also divides $a - bq = r$ (by Theorem 1 of Section 4.1). Hence, any common divisor of *a* and *b* must also be any common divisor of *b* and *r*.

– Suppose that *d* divides both *b* and *r*.

Then *d* also divides $bq + r = a$. Hence, any common divisor of *a* and *b* must also be a common divisor of *b* and *r*.

– Therefore, gcd(*a*,*b*) = gcd(*b*,*r*).

◀

Correctness of Euclidean Algorithm

- Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$. Successive applications of the division algorithm yields:

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 & 0 \leq r_2 < r_1,\\
r_1 &= r_2 q_2 + r_3 & 0 \leq r_3 < r_2,\\
&\ \vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_2 & 0 \leq r_n < r_{n-1},\\
r_{n-1} &= r_n q_n .
\end{aligned}
$$

- Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 > r_1 > r_2 > \cdots \geq 0$. The sequence can't contain more than a terms.

Correctness of Euclidean Algorithm

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&\ \vdots \\
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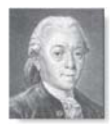
\end{aligned}
$$

- Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 > r_1 > r_2 > \cdots \geq 0$. The sequence can't contain more than a terms.

- By Lemma 1
  $\gcd(a,b) = \gcd(r_0,r_1) = \cdots = \gcd(r_{n-1},r_n) = \gcd(r_n, 0) = r_n$.
- Hence the greatest common divisor is the last nonzero remainder in the sequence of divisions.

gcds as Linear Combinations



Étienne Bézout (1730-1783)

Bézout's Theorem: If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa + tb$.

(*proof in exercises of Section* 5.2)

Definition: If a and b are positive integers, then integers s and t such that $\gcd(a,b) = sa + tb$ are called *Bézout coefficients* of a and b. The equation $\gcd(a,b) = sa + tb$ is called *Bézout's identity*.

- By Bézout's Theorem, the gcd of integers a and b can be expressed in the form $sa + tb$ where s and t are integers. This is a *linear combination* with integer coefficients of a and b.
  - gcd(6,14) =

gcds as Linear Combinations

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- By Bézout's Theorem, the gcd of integers a and b can be expressed in the form $sa + tb$ where s and t are integers. This is a *linear combination* with integer coefficients of a and b.
  - gcd(6,14) =

    $= 2$
    $= (-2)\cdot 6 + 1\cdot 14$

Finding gcds as Linear Combinations

Example: Express gcd(252,198) = 18 as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show gcd(252,198) = 18

- Now working backwards, from iii and i above
- Substituting the 2nd equation into the 1st yields:
- Substituting $54 = 252 - 1 \cdot 198$ (from i)) yields:

- This method illustrated above is a two pass method. It first uses the Euclidian algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers.

Finding gcds as Linear Combinations

Example: Express gcd(252,198) = 18 as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show gcd(252,198) = 18

i. $252 = 1 \cdot 198 + 54$
ii. $198 = 3 \cdot 54 + 36$
iii. $54 = 1 \cdot 36 + 18$
iv. $36 = 2 \cdot 18$

- Now working backwards, from iii and i above
  - $18 = 54 - 1 \cdot 36$
  - $36 = 198 - 3 \cdot 54$
- Substituting the 2nd equation into the 1st yields:
  - $18 = 54 - 1 \cdot (198 - 3 \cdot 54) = 4 \cdot 54 - 1 \cdot 198$
- Substituting $54 = 252 - 1 \cdot 198$ (from i)) yields:
  - $18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198$

- This method illustrated above is a two pass method. It first uses the Euclidian algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers.

Consequences of Bézout's Theorem

Lemma 2: If a, b, and c are positive integers such that gcd(a, b) = 1 and $a \mid bc$, then $a \mid c$.

Proof: Assume gcd(a, b) = 1 and $a \mid bc$

- Since gcd(a, b) = 1, by Bézout's Theorem there are integers s and t such that

$$sa + tb = 1.$$

Lemma 3: If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.

(*proof uses mathematical induction; see Exercise* 64 *of Section* 5.1)

- Lemma 3 is crucial in the proof of the uniqueness of prime factorizations.

Consequences of Bézout's Theorem

Lemma 2: If a, b, and c are positive integers such that gcd(a, b) = 1 and $a \mid bc$, then $a \mid c$.

Proof: Assume gcd(a, b) = 1 and $a \mid bc$

- Since gcd(a, b) = 1, by Bézout's Theorem there are integers s and t such that

$$sa + tb = 1.$$

- Multiplying both sides of the equation by c, yields $sac + tbc = c$.
- From Theorem 1 of Section 4.1:

  $a \mid tbc$ (part ii) and a divides $sac + tbc$ since $a \mid sac$ and $a \mid tbc$ (part i)
- We conclude $a \mid c$, since $sac + tbc = c$.

Lemma 3: If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.

(*proof uses mathematical induction; see Exercise* 64 *of Section* 5.1)

- Lemma 3 is crucial in the proof of the uniqueness of prime factorizations.

Uniqueness of Prime Factorization

- We will prove that a prime factorization of a positive integer where the primes are in nondecreasing order is unique. (This is part of the fundamental theorem of arithmetic. The other part, which asserts that every positive integer has a prime factorization into primes, will be proved in Section 5.2.)

  Proof: (*by contradiction*) Suppose that the positive integer n can be written as a product of primes in two distinct ways:

  $n = p_1 p_2 \cdots p_s$ and $n = q_1 q_2 \cdots p_t$.

  – Remove all common primes from the factorizations to get

  $$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v}.$$

  – By Lemma 3, it follows that p_{i_1} divides q_{j_k}, for some k, contradicting the assumption that p_{i_1} and q_{j_k} are distinct primes.

  – Hence, there can be at most one factorization of n into primes in nondecreasing order. ◀

Dividing Congruences by an Integer

- Dividing both sides of a valid congruence by an integer does not always produce a valid congruence (see Section 4.1).
- But dividing by an integer relatively prime to the modulus does produce a valid congruence:

  Theorem 7: Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc$ (mod m) and gcd(c,m) = 1, then $a \equiv b$ (mod m).

  Proof:

◀

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  Proof: Since $ac \equiv bc$ (mod m), $m \mid ac - bc = c(a - b)$ by Lemma 2 and the fact that gcd(c,m) = 1, it follows that $m \mid a - b$.

  Hence, $a \equiv b$ (mod m). ◀

Chap 4.4 - Linear Congruences

Definition: A congruence of the form

$ax \equiv b$(mod m),

where m is a positive integer, a and b are integers, and x is a variable, is called a *linear congruence*.

- The solutions to a linear congruence $ax \equiv b$(mod m) are all integers x that satisfy the congruence.

Definition: An integer $\bar{a}$ such that $\bar{a}a \equiv 1$(mod m) is said to be an *inverse* of a modulo m.

Example: What is the inverse of 3 modulo 7?

- One method of solving linear congruences makes use of an inverse $\bar{a}$, if it exists. Although we can not divide both sides of the congruence by a, we can multiply by $\bar{a}$ to solve for x.

Chap 4.4 - Linear Congruences

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Definition: An integer $\bar{a}$ such that $\bar{a}a \equiv 1(\bmod m)$ is said to be an *inverse* of a modulo m.

Example: What is the inverse of 3 modulo 7?

5 is an inverse of 3 modulo 7 since $5 \cdot 3 = 15 \equiv 1 (\bmod 7)$

- One method of solving linear congruences makes use of an inverse $\bar{a}$, if it exists. Although we can not divide both sides of the congruence by a, we can multiply by $\bar{a}$ to solve for x.

Inverse of *a* modulo *m*

- The following theorem guarantees that an inverse of a modulo m exists whenever a and m are relatively prime. Two integers a and b are relatively prime when $\gcd(a,b) = 1$.

Theorem 1: If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. Furthermore, this inverse is unique modulo m. (This means that there is a unique positive integer $\bar{a}$ less than m that is an inverse of a modulo m and every other inverse of a modulo m is congruent to $\bar{a}$ modulo m.)

Proof: Since $\gcd(a,m) = 1$, by Theorem 6 of Section 4.3, there are integers s and t such that $sa + tm = 1$.

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Proof: Since $\gcd(a,m) = 1$, by Theorem 6 of Section 4.3, there are integers s and t such that $sa + tm = 1$.

- Hence, $sa + tm \equiv 1 (\bmod m)$.
- Since $tm \equiv 0 (\bmod m)$, it follows that $sa \equiv 1 (\bmod m)$
- Consequently, s is an inverse of a modulo m.
- The uniqueness of the inverse is Exercise 7.

Finding Inverses

- The Euclidean algorithm and Bézout coefficients gives us a systematic approaches to finding inverses.

Example: Find an inverse of 3 modulo 7.

Solution: Because $\gcd(3,7) = 1$, by Theorem 1, an inverse of 3 modulo 7 exists.

- Using the Euclidian algorithm to find gcd: $7 = 2 \cdot 3 + 1$.
- From this equation, we get $-2 \cdot 3 + 1 \cdot 7 = 1$, and see that -2 and 1 are Bézout coefficients of 3 and 7.
- Hence, -2 is an inverse of 3 modulo 7.
- Also every integer congruent to -2 modulo 7 is an inverse of 3 modulo 7, i.e., 5, -9, 12, etc.

Finding Inverses

Example: Find an inverse of 101 modulo 4620.

Solution: First use the Euclidian algorithm to show that gcd(101,4620) = 1.

Working Backwards:

Finding Inverses

Example: Find an inverse of 101 modulo 4620.

Solution: First use the Euclidian algorithm to show that gcd(101,4620) = 1.

Working Backwards:

$4620 = 45 \cdot 101 + 75$
$101 = 1 \cdot 75 + 26$
$75 = 2 \cdot 26 + 23$
$26 = 1 \cdot 23 + 3$
$23 = 7 \cdot 3 + 2$
$3 = 1 \cdot 2 + 1$
$2 = 2 \cdot 1$

$1 = 3 - 1 \cdot 2$
$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$
$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$
$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$
$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$
$\quad = 26 \cdot 101 - 35 \cdot 75$
$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$
$\quad = -35 \cdot 4620 + 1601 \cdot 101$

Since the last nonzero remainder is 1, gcd(101,4620) = 1

Bézout coefficients : -35 and 1601

1601 is an inverse of 101 modulo 4620