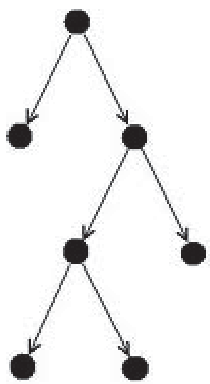
CompSci 102
Discrete Math for Computer Science



March 13, 2012

Prof. Rodger

Slides modified from Rosen

Announcements

- Read for next time Chap. 6.1-6.2
- Recitation on Friday
- Homework 5 out

Strong Induction

- *Strong Induction*: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:
  - *Basis Step*: Verify that the proposition $P(1)$ is true.
  - *Inductive Step*: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$ holds for all positive integers k.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Strong Induction and the Infinite Ladder

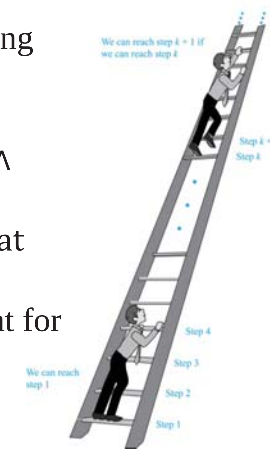
Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer k, if we can reach the first k rungs, then we can reach the $(k + 1)$st rung.

To conclude that we can reach every rung by strong induction:

- BASIS STEP: $P(1)$ holds
- INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer k, and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer n, $P(n)$ holds, i.e., we can reach the nth rung of the ladder.



Proof using Strong Induction

Example: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Prove that we can reach every rung.

(Try this with mathematical induction.)

Solution: Prove the result using strong induction.

- BASIS STEP: We can reach the first step.
- INDUCTIVE STEP: The inductive hypothesis is that we can reach the first k rungs, for any $k \geq 2$.
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Solution: Prove the result using strong induction.

- BASIS STEP: We can reach the first step.
- INDUCTIVE STEP: The inductive hypothesis is that we can reach the first k rungs, for any $k \geq 2$.
- We can reach the $(k+1)$st rung since we can reach the $(k-1)$st rung by the inductive hypothesis.
- Hence, we can reach all rungs of the ladder.

Which Form of Induction Should Be Used?

- Can always use strong induction instead of mathematical induction.
- (if it is simpler use mathematical induction).
- The principles of mathematical induction, strong induction, and the well-ordering property are all equivalent. (*Exercises* 41-43)
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Completion of the proof of the Fundamental Theorem of Arithmetic

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as a product of primes.

- BASIS STEP: $P(2)$ is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is $P(j)$ is true for all integers j with $2 \leq j \leq k$.
- To show that $P(k + 1)$ must be true under this assumption, two cases need to be considered:

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- To show that $P(k + 1)$ must be true under this assumption, two cases need to be considered:
  - If $k + 1$ is prime, then $P(k + 1)$ is true.
  - Otherwise, $k + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k + 1$ can also be written as the product of those primes.

Hence, it has been shown that every integer greater than 1 can be written as the product of primes. ◀

Proof using Strong Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses
  - $P(13)$ uses
  - $P(14)$ uses .
  - $P(15)$ uses
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
- Using the inductive hypothesis,
  To form postage of $k + 1$ cents,

Hence, $P(n)$ holds for all $n \geq 12$. ◀

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- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps.
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
  - $P(15)$ uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents.

Hence, $P(n)$ holds for all $n \geq 12$. ◀

Q: Why do we need 4 basis steps?

Proof of Same Example using Mathematical Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: Postage of 12 cents can be formed using
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show P($k + 1$) where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used,
  - Otherwise, no 4-cent stamp have been used

◀

Hence, $P(n)$ holds for all $n \geq 12$.

Proof of Same Example using Mathematical Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show P(k + 1) where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of k + 1 cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of k + 1 cents. ◀

Hence, $P(n)$ holds for all $n \geq 12$.

Well-Ordering Property

- *Well-ordering property*: Every nonempty set of nonnegative integers has a least element.
- The well-ordering property is one of the axioms of the positive integers listed in Appendix 1.
- The well-ordering property can be used directly in proofs, as the next example illustrates.
- The well-ordering property can be generalized.
  - **Definition:** A set is *well ordered if every subset has a least element.*
    - **N** is well ordered under ≤.
    - The set of finite strings over an alphabet using lexicographic ordering is well ordered.
  - We will see a generalization of induction to sets other than the integers.

Well-Ordering Property

Example: Use the well-ordering property to prove the division algorithm, which states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$, such that $a = dq + r$.

Solution: Let S be the set of nonnegative integers of the form $a - dq$, where q is an integer. The set is nonempty since $-dq$ can be made as large as needed.

- By the well-ordering property, S has a least element $r = a - dq_0$. The integer r is nonnegative. It also must be the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1) = a - dq_0 - d = r - d > 0$.
- Therefore, there are integers q and r with $0 \leq r < d$.

  (*uniqueness of q and r is Exercise* 37) ◀

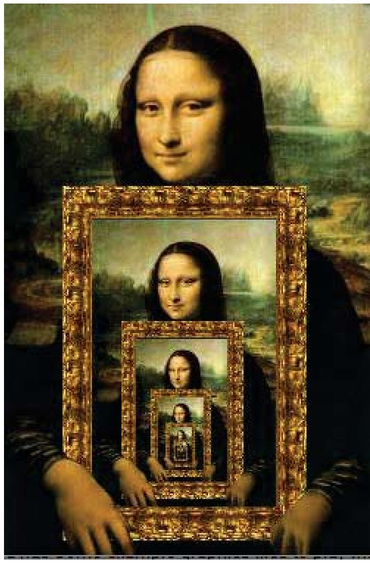
Sec 5.3 - Recursively Defined Functions

Definition: A *recursive* or *inductive definition* of a function consists of two steps.

- BASIS STEP: Specify the value of the function at zero.
- RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

- A function $f(n)$ is the same as a sequence a_0, a_1, … , where a_i, where $f(i) = a_i$. This was done using recurrence relations in Section 2.4.

Recursively Defined Pictures



Recursively Defined Functions

Example: Suppose f is defined by:

$f(0) = 3,$

$f(n + 1) = 2f(n) + 3$

Find $f(1), f(2), f(3), f(4)$

Solution:

- $f(1) =$
- $f(2) =$
- $f(3) =$
- $f(4) =$

Example: Give a recursive definition of the factorial function $n!$:

Solution:

Recursively Defined Functions

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Find $f(1), f(2), f(3), f(4)$

Solution:

- $f(1) = 2f(0) + 3 = 2\cdot3 + 3 = 9$
- $f(2) = 2f(1)+ 3 = 2\cdot9 + 3 = 21$
- $f(3) = 2f(2) + 3 = 2\cdot21 + 3 = 45$
- $f(4) = 2f(3) + 3 = 2\cdot45 + 3 = 93$

Example: Give a recursive definition of the factorial function $n!$:

Solution:

$f(0) = 1$

$f(n + 1) = (n + 1)\cdot f(n)$

Recursively Defined Functions

Example: Give a recursive definition of:

$$\sum_{k=0}^{n} a_k.$$

Solution: The first part of the definition is

The second part is

Recursively Defined Functions

Example: Give a recursive definition of:

$$\sum_{k=0}^{n} a_k.$$

Solution: The first part of the definition is

$$\sum_{k=0}^{0} a_k = a_0.$$

The second part is $$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}.$$

Fibonacci Numbers

Fibonacci
(1170- 1250)

Example : The Fibonacci numbers are defined as follows:

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

Find f_2, f_3, f_4, f_5.

- $f_2 = f_1 + f_0 = 1 + 0 = 1$
- $f_3 = f_2 + f_1 = 1 + 1 = 2$
- $f_4 = f_3 + f_2 = 2 + 1 = 3$
- $f_5 = f_4 + f_3 = 3 + 2 = 5$

Fibonacci Numbers

Example 4:

Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where $\alpha = (1 + \sqrt{5})/2$.

Solution: Let $P(n)$ be the statement $f_n > \alpha^{n-2}$. Use strong induction to show that $P(n)$ is true whenever $n \geq 3$.

- BASIS STEP: $P(3)$ holds since $\alpha < 2 = f_3$

  $P(4)$ holds since $\alpha^2 = (3 + \sqrt{5})/2 < 3 = f_4$.
- INDUCTIVE STEP: Assume that $P(j)$ holds, i.e., $f_j > \alpha^{j-2}$ for all integers j with

  $3 \leq j \leq k$, where $k \geq 4$. Show that $P(k + 1)$ holds, i.e., $f_{k+1} > \alpha^{k-1}$.
  - Since $\alpha^2 = \alpha + 1$ (because α is a solution of $x^2 - x - 1 = 0$),
  - By the inductive hypothesis, because $k \geq 4$ we have
  - Therefore, it follows that
  - Hence, $P(k + 1)$ is true.

Fibonacci Numbers

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  $3 \leq j \leq k$, where $k \geq 4$. Show that $P(k + 1)$ holds, i.e., $f_{k+1} > \alpha^{k-1}$.
  - Since $\alpha^2 = \alpha + 1$ (because α is a solution of $x^2 - x - 1 = 0$),

$$\alpha^{k-1} = \alpha^2 \cdot \alpha^{k-3} = (\alpha + 1) \cdot \alpha^{k-3} = \alpha \cdot \alpha^{k-3} + 1 \cdot \alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$$

  - By the inductive hypothesis, because $k \geq 4$ we have

$$f_{k-1} > \alpha^{k-3}, \qquad f_k > \alpha^{k-2}.$$

  - Therefore, it follows that

$$f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-1}.$$

  - Hence, $P(k + 1)$ is true.