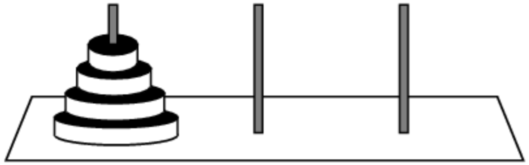
CompSci 102
Discrete Math for Computer Science



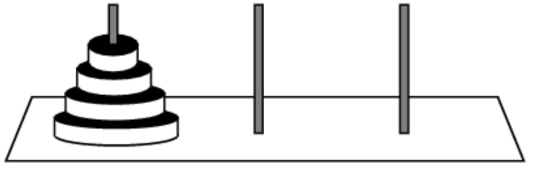
April 17, 2012

Prof. Rodger

Announcements

- Read for next time Chap. 8.3-8.4, 10.1-10.2
- Last Recitation on Friday
- Test back today

Recurrence Relations
- model lots of problems



- The Tower of Hanoi
- Divide and conquer algorithms
  - Sorting algorithm mergesort
  - Sorting algorithm quicksort
- Tree algorithms
  - Searching for an element in a binary search tree
  - Listing out all elements in a binary search tree

Solving a recurrence relation

- Problem sets up as a recurrence
  - Must have a base case

- Solve the recurrence
  - Use substitution
- Prove correctness
  - Proof by induction

Example 1

- $a_n = a_{n-1} + c$
- $a_0 = 1$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Example 1

$a_0 = 1$

$a_n = a_{n-1} + c$

$a_n = [a_{n-2} + c] + c$ substitute $a_{n-1} = a_{n-2} + c$

$= a_{n-2} + 2c$

$= [a_{n-3} + c] + 2c$

$= a_{n-3} + 3c$

…

$= a_{n-k} + kc$

… let $n-k = 0 \rightarrow k = n$

$= a_0 + nc = cn + 1$

Proof by induction

Basis: $a_0 = 1$

$a_n = cn + 1 = c(0) + 1 = 1$ ✓

Assume: $a_k = ck + 1$ for all $k < n$

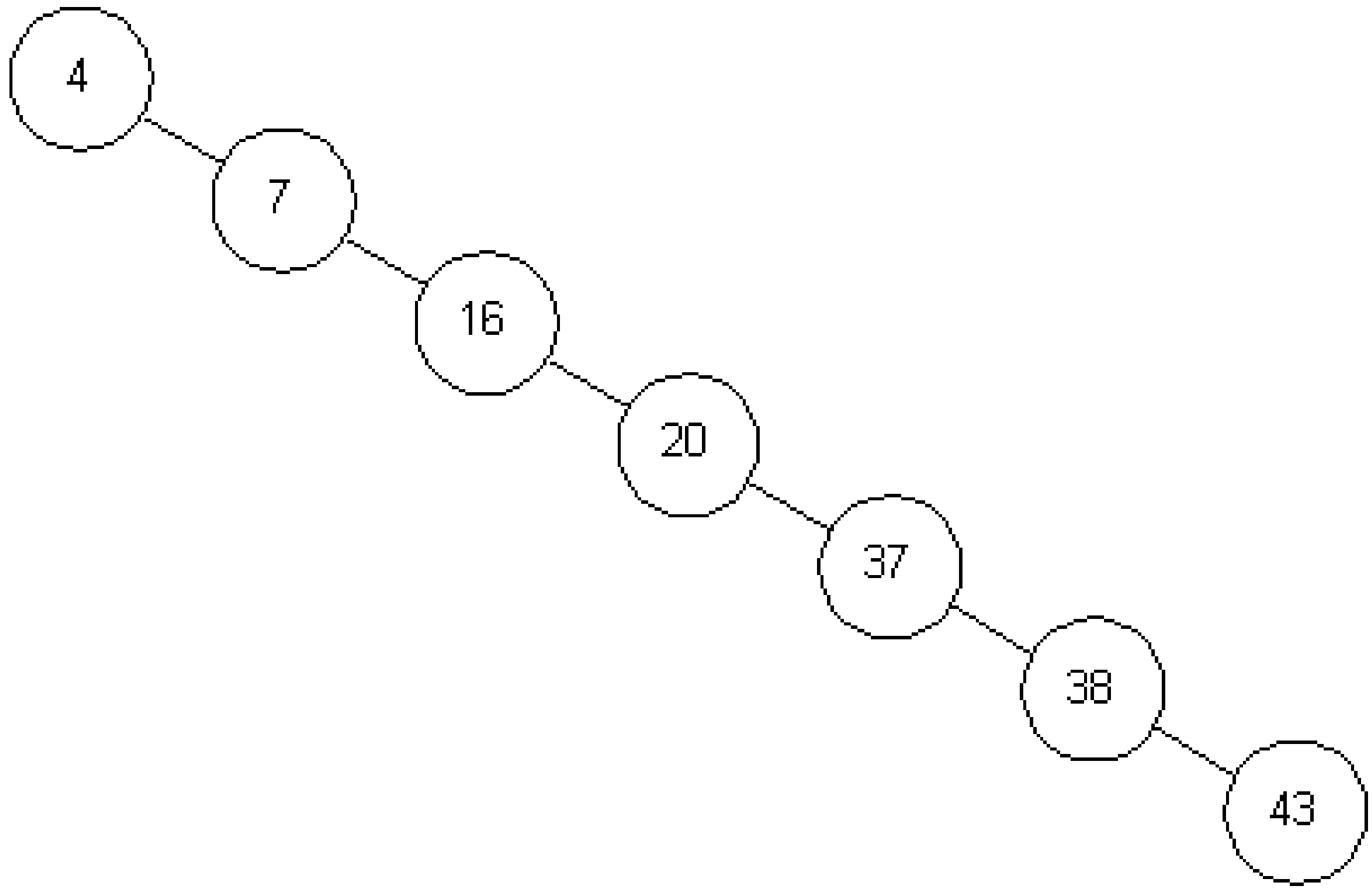
$a_n = a_{n-1} + c$

$= [c(n-1) + 1] + c$ by I.H.

$= cn – c + 1 + c$

$= cn + 1$ ✓

Worst case binary search tree



Example 2

- $a_n = 2*a_{n-1} + c$
- $a_0 = 0$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Example 2

$a_0 = 0$

$a_n = 2 * a_{n-1} + c$

$= 2 * [2 * a_{n-2} + c] + c$

$= 2^2\ a_{n-2} + 2c + c$

$= 2^2 * [2 * a_{n-3} + c] + 2c + c$

$= 2^3\ a_{n-3} + 4c + 2c + c$

…

$= 2^k\ a_{n-k} + c(2^{k-1} + \ldots + 2 + 1)$

$= 2^k\ a_{n-k} + c(2^k - 1)$

Example 2

$= 2^k\ a_{n-k}\ + c(2^k\ - 1)$

…

let n-k = 0 → k = n

$= 2^n\ a_0 + c(2^n\ - 1)$

$= 2^n\ *(0) + c(2^n\ - 1)$

$= c\ (2^n\ - 1)$

Note: this is exponential!

Proof by Induction

Basis: $a_0 = 0$

$$a_0 = c(2^0 - 1) = c(1 - 1) = 0 \quad ✓$$

Assume: $a_k = c\,(2^k - 1)$ for all $k < n$

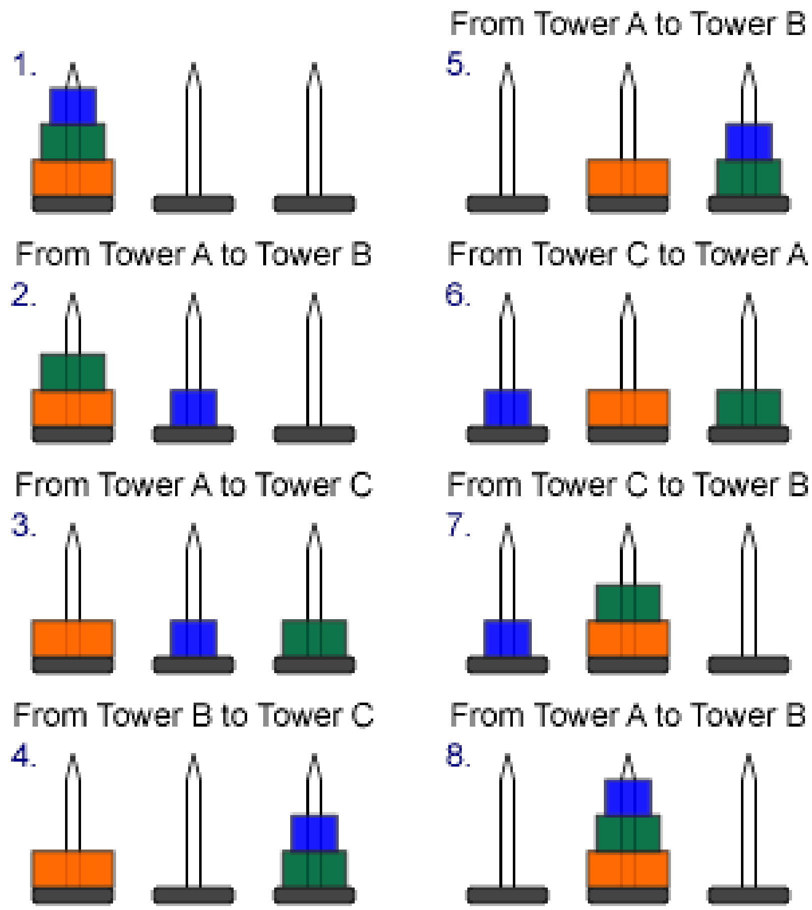
$$a_n = 2 * a_{n-1} + c$$

$$= 2 * [c(2^{n-1} - 1)] + c \quad \text{by I.H.}$$

$$= c2^n - 2c + c = c(2^n - 1) \quad ✓$$

Towers of Hanoi

- Figures
  - Figs 1-4
    - problem size n-1
  - Figs 4-5
    - Constant work
  - Figs 5-7
    - problem size n-1



Example 3

- $a_n = 2*a_{n/2} + c$
- $a_1 = c$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Example 3

$a_1 = c$

$a_n = 2*a_{n/2} + c$

$= 2*[2 * a_{n/4} + c] + c$

$= 2^2 * a_{n/4} + 2c + c$

$= 2^2 * [2 * a_{n/8} + c] + 2c + c$

$= 2^3 * a_{n/8} + 4c + 2c + c$

…

$= 2^k * a_{n/2^k} + c(2^k - 1)$

Let $n = 2^k$

Example 3

$= n * a_1 + c(n – 1)$

$= n * (c) + cn – c$

$= 2cn – c$

$a_n = 2cn - c$

Prove by induction

Basis: $a_1 = c$

$a_n = 2cn – c$

$a_1 = 2c(1) – c = c$ ✓

Assume: $a_k = 2 c k - c$

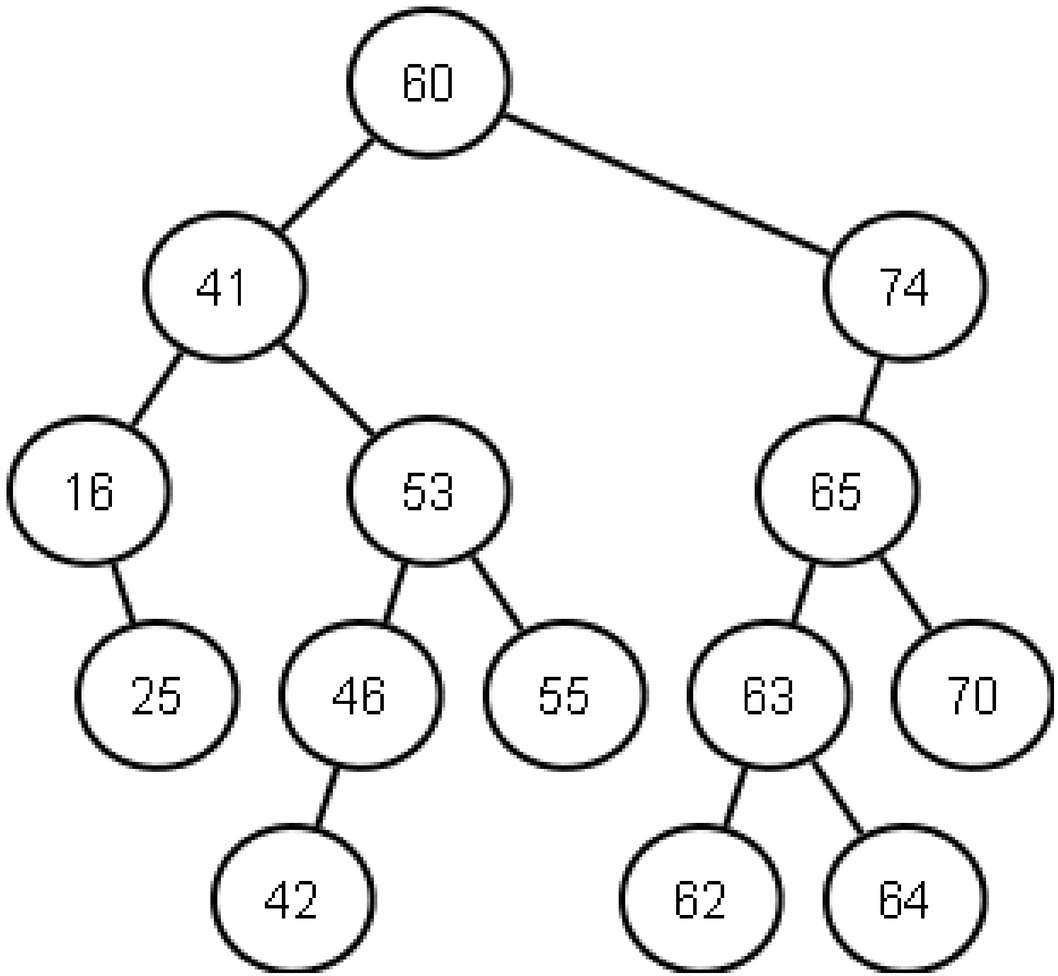
$a_n = 2 * a_{n/2} + c$

$= 2 * [2cn/2 – c] + c$

$= 2cn -2c + c$

$= 2cn - c$ ✓

Traversal in binary search tree preorder, postorder, inorder



Example 4

- $a_n = 2*a_{n/2} + cn$
- $a_1 = c$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Example 4

$a_1 = c$

$a_n = 2*a_{n/2} + cn$

$= 2*[2*a_{n/4} + cn/2] + cn$

$= 2^2\ a_{n/4} + 2cn$

$= 2^2\ [2*a_{n/8} + cn/4] + 2cn$

$= 2^3 * a_{n/8} + 3cn$

…

$= 2^k * a_{n/2^k} + kcn$

Let $n/2^k = 1 \rightarrow k = \log_2 n$

Example 4

- $= n * a_1 + (\log_2 n)cn$
- $= nc + cn\log_2 n$
- $= cn(1 + \log_2 n)$

Proof by induction

Basis: $a_1 = c$

$$a_1 = cn(1 + \log_2 n)$$

$$= c(1)(1 + \log_2 1)$$

$$= c * (1 + 0) = c \qquad \checkmark$$

Assume: $a_k = ck + ck \log_2 k \quad$ for all $k<n$

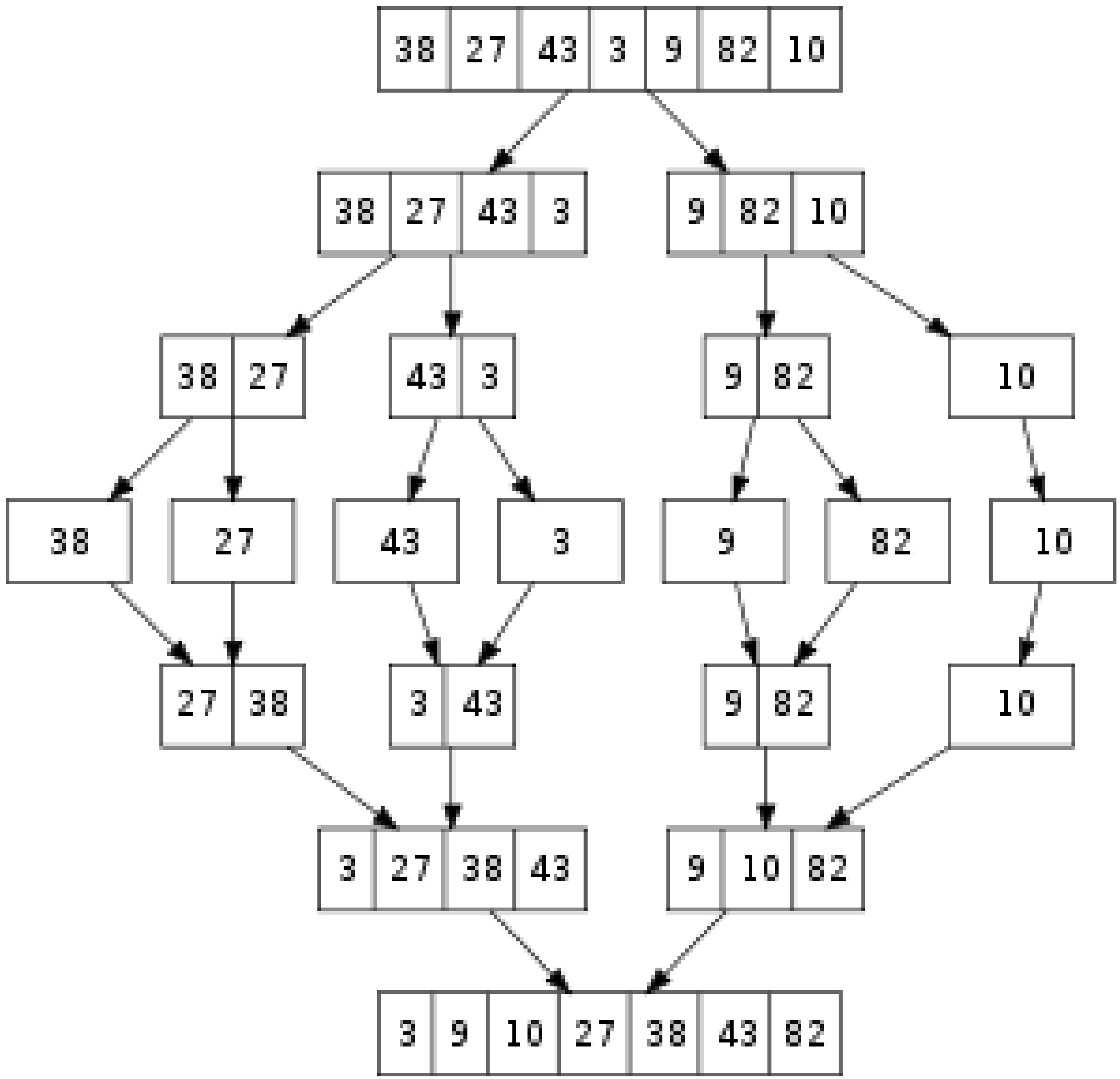
$$a_n = 2 * a_{n/2} + cn$$

$$= 2 * [cn/2 + cn/2 \log_2 n/2] + cn$$

$$= cn + cn (\log_2 n + \log_2 ½) + cn$$

$$= cn(2 + \log_2 n – 1) = cn(1 + \log_2 n) \qquad \checkmark$$

MergeSort



- n log n

Definition

- A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form
- $a_n = c_1 a_{n-1} + \ c_2 a_{n-2} + \ \ldots + \ c_k a_{n-k}$
- Where c_i are real numbers and $c_k \neq 0$

8.2 - Theorem 1

- Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_1^n + \propto_2 r_2^n$$

for all n where $\propto_1$ and $\propto_2$ are constants

Example

- What is the solution to the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$ with $a_0 = 2, a_1 = 7$?

Example

- What is the solution to the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$ with $a_0 = 2, a_1 = 7$ *?*
- $r^2 - c_1 r - c_2 = 0$

  $r^2 - r - 2 = 0 \;\; \rightarrow \;\; r_1 = 2, r_2 = -1$
- $a_n = \alpha_1 r_1 + \alpha_2 r_2$
- $a_n = \alpha_1 2^n + \propto_2 -1^n$
- $2 = a_0 = \alpha_1 \cdot 1 + \alpha_2 \cdot 1$
- $7 = a_1 = \alpha_1 \cdot 2 + \alpha_2 \cdot -1$
- $\rightarrow 9 = 3\, \alpha_1,\; \alpha_1 = 3, \alpha_2 = -1$
- $a_n = 3 \cdot 2^n - (-1)^n$

8.2 - Theorem 2

- Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_0^n + \propto_2 n\, r_0^n$$

for all n where $\propto_1$ and $\propto_2$ are constants

Many other theorems

- See theorems 2-6 in Chapter 8.2

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Theorem 1 in 8.3

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a > 1, \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2,$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Master Theorem in 8.3

MASTER THEOREM Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$