CompSci 102
Discrete Math for Computer Science

April 17, 2012

Prof. Rodger

Announcements

- Read for next time Chap. 8.3-8.4, 10.1-10.2
- Last Recitation on Friday
- Test back today

Recurrence Relations
- model lots of problems

- The Tower of Hanoi
- Divide and conquer algorithms
  - Sorting algorithm mergesort
  - Sorting algorithm quicksort
- Tree algorithms
  - Searching for an element in a binary search tree
  - Listing out all elements in a binary search tree

Solving a recurrence relation

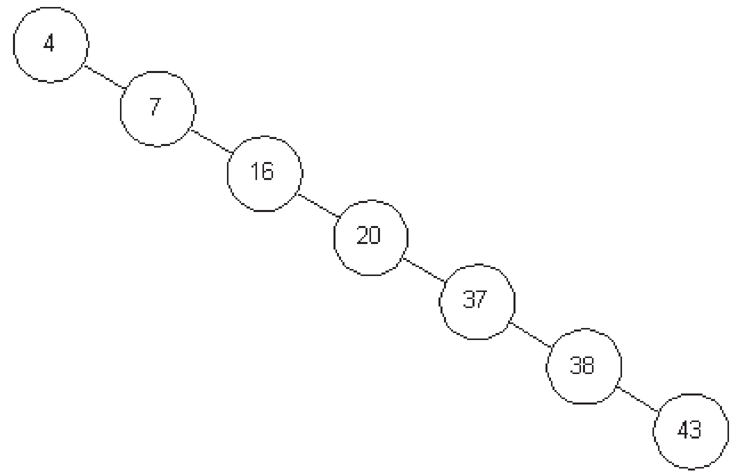
- Problem sets up as a recurrence
  - Must have a base case
- Solve the recurrence
  - Use substitution
- Prove correctness
  - Proof by induction

Example 1

- $a_n = a_{n-1} + c$
- $a_0 = 1$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Worst case binary search tree



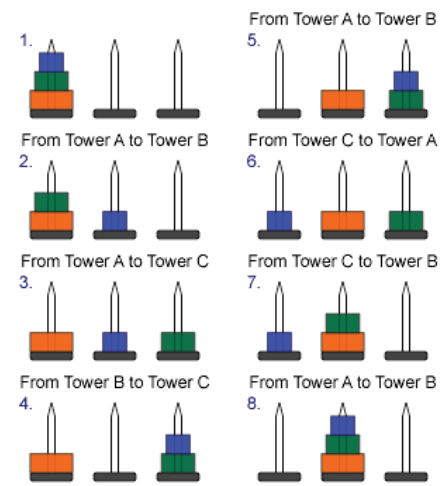
Example 2

- $a_n = 2*a_{n-1} + c$
- $a_0 = 0$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Towers of Hanoi

- Figures
  - Figs 1-4
    - problem size n-1
  - Figs 4-5
    - Constant work
  - Figs 5-7
    - problem size n-1

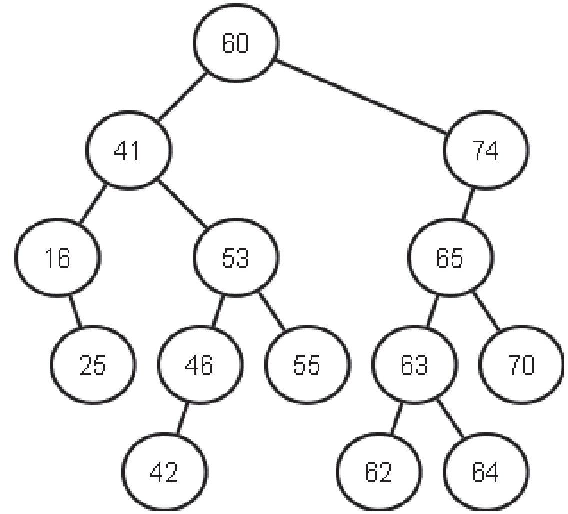


Example 3

- $a_n = 2*a_{n/2} + c$
- $a_1 = c$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Traversal in binary search tree preorder, postorder, inorder

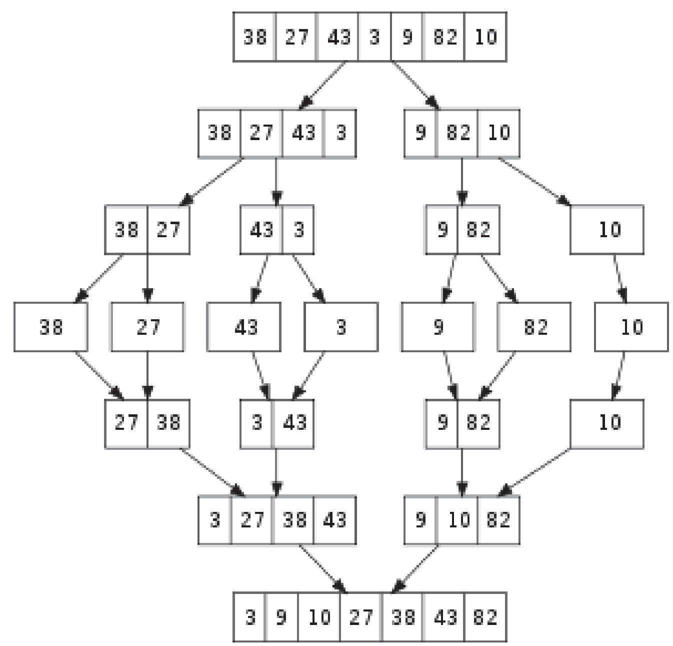


Example 4

- $a_n = 2*a_{n/2} + cn$
- $a_1 = c$

- Solve recurrence
- Then prove true by induction
- What is this an example of?

MergeSort



- n log n

Definition

- A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form
- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$
- Where c_i are real numbers and $c_k \neq 0$

Theorem

- Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_1^n + \propto_2 r_2^n$$

for all n where $\propto_1$ and $\propto_2$ are constants

Example

- What is the solution to the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$

Many other theorems

- See theorems 2-6 in Chapter 8.2

Theorem 1 in 8.3

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a > 1, \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2,$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Master Theorem in 8.3

MASTER THEOREM Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$