CPS 102 **Exam 2 Solutions**

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Dr. Rodger

1. (9 pts) Prime Numbers

   (a) What is the greatest common divisor of 24 and 84?
   ANSWER:
   $84 = 2^2 * 3 * 7$
   $24 = 2^3 * 3$
   $GCD = 2^2 * 3 = 12$

   (b) What is the least common multiple of 24 and 84?
   ANSWER:
   $84 = 2^2 * 3 * 7$
   $24 = 2^3 * 3$
   LCD $= 2^3 * 3 * 7 = 168$

   (c) Are the numbers 15, 16 and 21 *pairwise relatively prime*? Explain.
   ANSWER:
   The gcd(15,16)=1, gcd(16,21)=1, gcd(15,21)=3. Not pairwise relatively prime. To be all pairs would have to have gcd of 1, and one pair has a gcd of 3.

2. (6 pts) Express the gcd(270,660) as a linear combination of 270 and 660 by first finding the gcd using the Euclidean algorithm and then solving for the Bezout coefficients.

   ANSWER:

   660 = 2 * 270 + 120

   270 = 2 * 120 + 30

   120 = 4 * 30

   The gcd(270,660) = 30.

   30 = 270 - 2 * 120

   120 = 660 - 2 * 270

   Thus 30 = 270 - 2 * [660 - 2 * 270] by substitution

   30 = 5 * 270 - 2 * 660

3. (6 pts) Sarah has created a check digit for her order numbers. The order numbers are 6 digits total with the check digit being the 6th digit. The check digit is the sum of the first five digits mod 6.

(a) What is the 6th digit if the first 5 digits are 42321?

ANSWER:

12 mod 6 = 0

(b) The 3rd digit is smudged and unreadable, shown here as a Q in 31Q983. If the other numbers are correct, what could Q be?

ANSWER:

(3 + 1 + x + 9 + 8) mod 6 = 3

(21 + x) mod 6 = 3

x = 0 or x = 6

4. (4 pts) A **variant** of the Caeser cipher assigns digits to letters in the following manner 0 to A, 1 to B, etc., as shown below, and then uses f(p) = (p + 5) mod 26 as the encryption method.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

The following message was BFXM encrypted using this variant. What is the original message?

ANSWER:

B = 1 - 5 is 22 which is W

F is 5 - 5 is 0 which is A

X is 23 - 5 is 18 which is S

M is 12 - 5 is 7 which is H

WASH

5. (5 pts) Prove by mathematical induction $1/2 + 1/4 + 1/8 + \ldots + 1/2^n = (2^n - 1)/2^n$ for $n \geq 1$.

ANSWER:

Basis: (n=1) $(2^1 - 1)/2^1 = 1/2 = 1/2$ Check

Assume true for k, $1/2 + 1/4 + 1/8 + \ldots + 1/2^k = (2^k - 1)/2^k$

Show true for k+1

$1/2 + 1/4 + 1/8 + \ldots + 1/2^k + 1/2^{k+1}$

$= (2^k - 1)/2^k + 1/2^{k+1}$ by I.H.

$= 2 * (2^k - 1)/2 * 2^k + 1/2^{k+1}$

$= (2^{k+1} - 2)/2^{k+1} + 1/2^{k+1}$

$= (2^{k+1} - 1)/2^{k+1}$

6. (5 pts) Suppose that the only currency was \$3 bills and \$10 bills. Show that every dollar amount greater than \$17 could be made with these two types of bills.

   ANSWER:

   Basis: 6 \$3 bills will compute \$18

   1 \$10 bill and 3 \$3 bills computes \$19

   2 \$10 bills computes \$20

   Suppose k dollars can be formed with \$3 and \$10 bills for $k > 20$.

   Look at the amount for $k + 1$. Subtract \$1. By I.H. the k amount can be formed with \$3 and \$10 bills. Suppose there are two \$10 bills. Then for $k + 1$ they can be replaced with 7 \$3 bills. Otherwise there must be at least 3 \$3 bills. Those plus one more dollar can be replaced with a \$10 bill to make the $k + 1$ amount.

7. (3 pts) Suppose Imelda will wear one of her 4 hats, one of her 10 dresses and one of her 20 pairs of shoes. How many different outfits can she wear?

   ANSWER:

   4 * 10 * 20 = 800

8. (4 pts) How many bit strings of length 8 contain

   (a) exactly two 1's?

   ANSWER:

   [8 2] = 8!/6!*2! = 8*7/2 = 28

   (b) at most 6 1's?

   ANSWER:

   Total choices 2^8 - exactly 8 1's and exactly 7 1's

   2^8 - [8 8] - [8 7]

   2^8 - 1 - 8 = 2^8 - 9 = 256-9 = 247

9. (6 pts) How many strings are there of six uppercase letters A-Z such that

   (a) letters cannot be repeated?

   ANSWER:

   26*25*24*23*22*21

   (b) the letter C is there at least once and the letters X,Y,Z cannot be used?

   ANSWER:

   Number of possibilities with all letters minus the number of possibilities with the letter C there exactly once

   $23^6 - 22^6$

(c) at most one of the six letters is a vowel (vowels are A, E, I, O, U) and the string must start with S?

ANSWER:

no vowels + exactly 1 vowel

21^5 + 5 places vowel could be * 5 choices * 21^4 other choices

$= 21^5 + 25 * 21^4$

10. (4 pts) What is the value of the following?

(a)

$$\binom{5}{0}$$

ANSWER: 1

(b)

$$\binom{7}{3}$$

ANSWER: 7*6*5/3*2 = 35

11. (4 pts) Consider the string GOOGOL?

(a) How many different strings of length 6 can be made using the letters in GOOGOL?

ANSWER:

6!/(3!*2!) = 60

(b) How many different strings of length 6 can be made using the letters in GOOGOL if the three O's must be adjacent to each other?

ANSWER:

4!/2! = 12

12. (3 pts) What is the coefficient of x^3y^6 in $(x + y)^9$?

ANSWER:

[9 3] = 9!/3!*6! = 9*8*7/3*2 = 3*4*7 = 3*28 = 84

13. (6 pts) Cats are different from each other. There is one white cat and one black cat among 9 cats.

(a) How many different ways are there to arrange 4 cats in a line from the 9 cats if the white cat must be included?

ANSWER:

4 positions for white cat * 8 * 7 * 6

OR [8 3] * 4! = 1344

(b) How many different ways are there to arrange 5 cats in a line from the 9 cats if both the white cat and black cat must be included?

ANSWER:

5*4* 7 * 6 * 5

OR [7 3] * 5! = 4200

14. (3 pts) What is the probability that a positive integer greater than 0 and less than 100 picked at random has all distinct digits?

ANSWER:

90/99 = 10/11

15. (3 pts) Which is more likely: rolling a total of 5 when two dice are rolled or rolling a total of 5 when three dice are rolled? Explain.

ANSWER:

5 with 2 dice - 4 chances 6^2 4/36

5 with 3 dice - 6 chances 6^3 6/216

More likely with two dice since 4/36 > 6/216

16. (4 pts) A lottery is defined as follows. A player selects 6 different numbers out of the numbers 1 through 40. Then the lottery computer randomly selects 6 different numbers from the numbers 1 through 40. The order the numbers are selected does not matter. What is the probability that at least 5 of the 6 numbers the player selected are from the 6 numbers the computer selected?

ANSWER:

Player must select [6 5] or 6 ways to choose 5 of 6 numbers to match.

and [34 1] ways for the last number chosen not to match.

There are 1 + 6 * 34 winning tickets = 205

There are [40 6] tickets in all

probability is 205/[40 6]

ALSO WRITTEN AS ([6 5] * 34 + [6 6])/ [40 6]