

Friday, Apr. 13

1. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
 - (a) The first 13 letters of the permutation are in alphabetical order.
 - (b) a and z are next to each other in the permutation.
 - (c) a and z are separated by at least 23 letters in the permutation.
2. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.

3. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?
4. Assume that the probability that a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has
- (a) exactly three boys?
 - (b) at least one boy?
 - (c) at least one girl?
 - (d) all children of the same sex?

5. A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers yes.

(a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

(b) Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

6. Suppose that Ann selects a ball by first picking one of two boxes at random and then selecting a ball from this box. The first box contains three orange balls and four black balls, and the second box contains five orange balls and six black balls. What is the probability that Ann picked a ball from the second box if she has selected an orange ball?

(HINT: Use Bayes Theorem)

7. Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.
- (a) What is the probability that someone who tests positive has the genetic disease?
- (b) What is the probability that someone who tests negative does not have the disease?
8. What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \dots, 50\}$ and the purchaser wins nothing otherwise?
9. Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?

10. What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?
11. Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.
- (a) Find the probability that a particular ball lands in a specified bin.
 - (b) What is the expected number of balls that land in a particular bin?
 - (c) What is the expected number of balls tossed until a particular bin contains a ball?

(d) What is the expected number of balls tossed until all bins contain a ball?

(Hint: Let X_i denote the number of tosses required to have a ball land in the i th bin once $i-1$ bins contain a ball. Find $E(X_i)$ and use the linearity of expectations.)

12. In 2010, the puzzle designer Gary Foshee posed this problem: "Mr. Smith has two children, one of whom is a son born on a Tuesday. What is the probability that Mr. Smith has two sons?"

Show that there are two different answers to this puzzle, depending on whether Mr. Smith specifically mentioned his son because he was born on a Tuesday or whether he randomly chose a child and reported its gender and birth day of the week.

(HINT: For the first possibility, enumerate all the equally likely possibilities for the gender and birth day of the week of the other child. To do this, consider first the cases where the older child is a boy born on a Tuesday, and then the case where the older boy is not a boy born on a Tuesday.)