Due Date: Thursday, April 12, 2012

1 Divide and Conquer (20 points)

The Hadamard matrices $H_0, H_1, H_2 \ldots$ are defined as follows:

- H_0 is the 1×1 matrix $[1]$
- For $k > 0$, H_k is the $2k \times 2k$ matrix defined as:

$$\begin{pmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{pmatrix}$$

Show that if v is a column vector of length $n = 2k$, then the matrix-vector product $H_k v$ can be calculated using $O(n \log n)$ operations. Assume that all the numbers involved are small enough that basic arithmetic operations like addition and multiplication take unit time.

2 Revisiting shortest paths, DPV 4.11 (20 points)

Give an algorithm that takes as input a directed graph with positive edge lengths, and returns the length of the shortest cycle in the graph (if the graph is acyclic, it should say so). Your algorithm should take time at most $O(V^3)$.

3 Spanning Trees of a graph, DPV 5.10 (20 points)

Let T be an MST of a graph G. Given a connected subgraph H of G, show that $T \cap H$ is contained in some MST of H.

4 Linear Programming, DPV 7.13 (20 points)

Consider the following simple 2-player game. The players (call them R and C) each choose an outcome, *heads* or *tails*. If both outcomes are equal, C gives a dollar to R; if the outcomes are different, R gives a collar to C.

- Represent the payoffs by a 2×2 matrix.
- What is the value of this game, and what are the optimal strategies for the two players. Write the LP for each case and describe the optimal solution.

5 Linear Programming, DPV 7.12 (20 points)

For the linear program

$$\max x_1 - 2x_3$$

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ 2x_2 - x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

prove that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal.

6 Linear Programming, DPV 7.27 (20 points)

Given an unlimited supply of coins of denominations $x_1, x_2, \ldots x_n$, we wish to make change for a value v; that is, we wish to find a set of coins whose total value is v. Show that this problem can be formulated as an integer linear program. Can we solve this program as an LP, in the certainty that the solution will turn out to be integral (as in the case of bipartite matching)? Either prove it or give a counterexample.