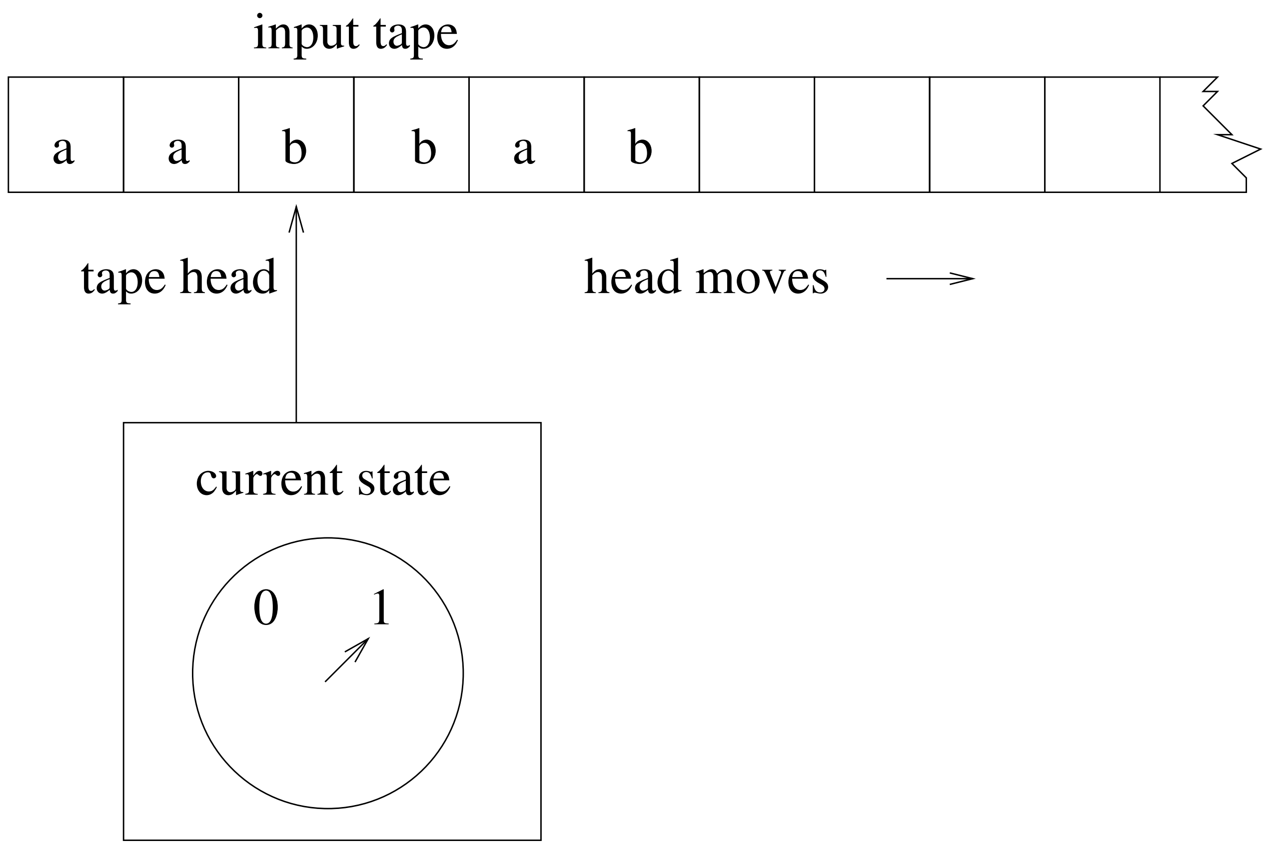
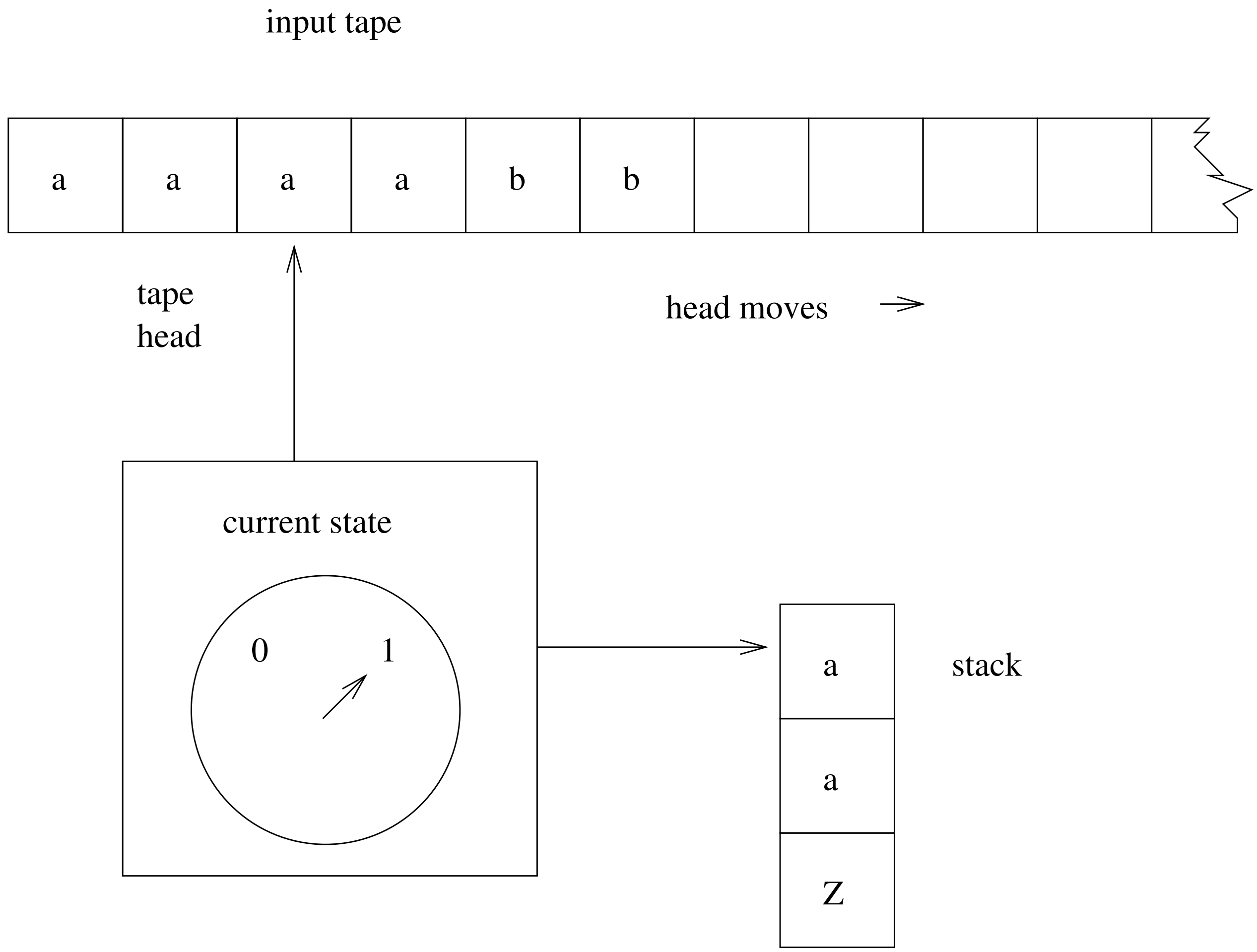
Section: Pushdown Automata

Ch. 7 - Pushdown Automata

A DFA=(Q,Σ,δ,q_0,F)



Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).



Definition: Nondeterministic PDA (NPDA) is defined by

$$\mathbf{M}=(\mathbf{Q},\Sigma,\ \Gamma,\ \delta,\ q_0,\ \mathbf{z},\ \mathbf{F})$$

where

Q is finite set of states
Σ **is tape (input) alphabet**
Γ **is stack alphabet**
q_0 **is initial state**
z - start stack symbol, z $\in \Gamma$
F $\subseteq$ **Q is set of final states.**
δ**:Q**$\times(\Sigma \cup \{\lambda\})\times\Gamma\rightarrow$ **finite subsets of** $Q \times \Gamma^*$

Example of transitions

$\delta(q_1,\mathbf{a},\mathbf{b}) = \{(q_3,\mathbf{b}),(q_4,\mathbf{ab}),(q_6,\lambda)\}$

The diagram for the above transitions is:

Instantaneous Description:

$$(\mathbf{q},\mathbf{w},\mathbf{u})$$

Description of a Move:

$$(q_1,\mathbf{aw},\mathbf{bx}) \vdash (q_2,\mathbf{w},\mathbf{yx})$$

iff

Definition Let M=(Q,Σ,Γ,δ,q_0,z,F) be a NPDA. L(M)=$\{$w$\in \Sigma^*$ $\mid$ (q_0,w,z) $\overset{*}{\vdash}$ (p,λ,u), p$\in$F, u$\in \Gamma^*\}$. The NPDA accepts all strings that start in q_0 and end in a final state.

Example: L=$\{a^n b^n | n \geq 0\}$**,** $\Sigma = \{a, b\}$**,**
$\Gamma = \{z, a\}$

Another Definition for Language Acceptance

NPDA M accepts L(M) by empty stack:

$$\mathbf{L(M)}=\{w \in \Sigma^* | (q_0, w, z) \stackrel{*}{\vdash} (p, \lambda, \lambda)\}$$

Example: L=$\{a^n b^m c^{n+m} | n, m > 0\}$**,**
$\Sigma = \{a, b, c\}$**,** $\Gamma = \{0, z\}$

Example: L=$\{ww^R | w \in \Sigma^+\}$**,** $\Sigma = \{a, b\}$**,** $\Gamma = \{z, a, b\}$

Example: L=$\{ww|w \in \Sigma^*\}$**,** $\Sigma = \{a, b\}$

Examples for you to try on your own: (solutions are at the end of the handout).

- **L=**$\{a^n b^m | m > n, m, n > 0\}$**,** $\Sigma = \{a, b\}$**,** $\Gamma = \{z, a\}$
- **L=**$\{a^n b^{n+m} c^m | n, m > 0\}$**,** $\Sigma = \{a, b, c\}$**,**
- **L=**$\{a^n b^{2n} | n > 0\}$**,** $\Sigma = \{a, b\}$

Definition: A PDA M=(Q,Σ,Γ,δ,q_0,z,F) is *deterministic* **if for every** $q \in$**Q,** $a \in \Sigma \cup \{\lambda\}$**,** $b \in \Gamma$

1. $\delta(q, a, b)$ **contains at most 1 element**
2. **if** $\delta(q, \lambda, b) \neq \emptyset$ **then** $\delta(q, c, b)$=$\emptyset$ **for all** $c \in \Sigma$

Definition: L is DCFL iff $\exists$ **DPDA M s.t. L=L(M).**

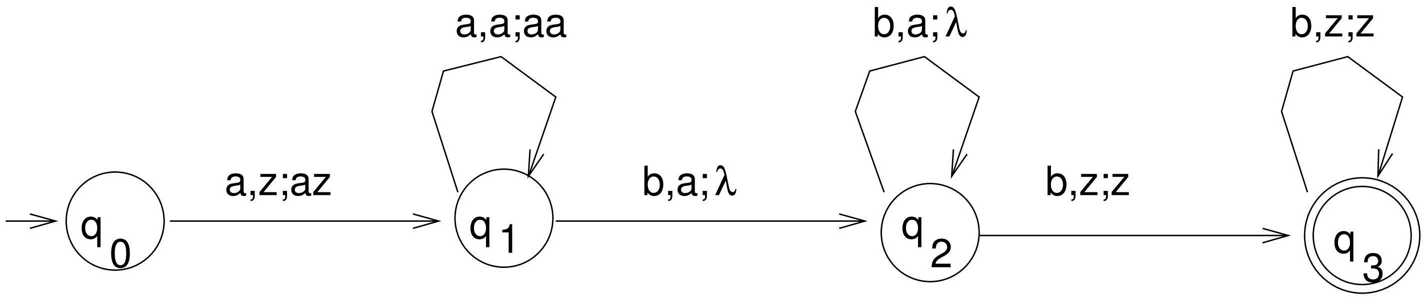
Examples:

1. **Previous pda for** $\{a^n b^n | n \geq 0\}$ **is deterministic?**

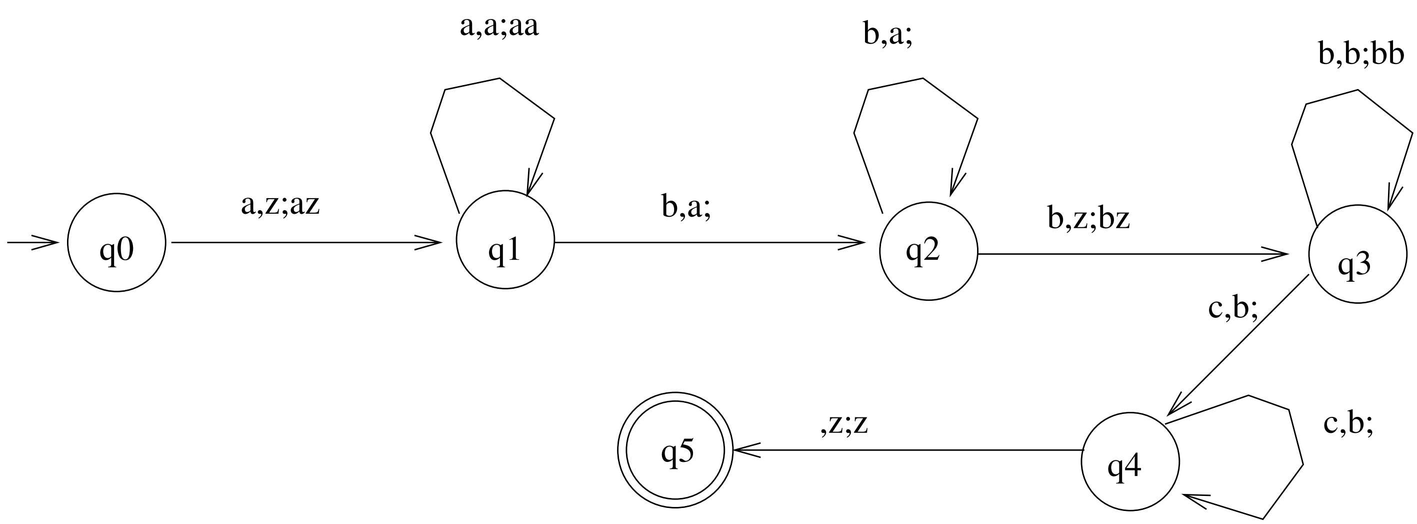
2. **Previous pda for** $\{a^n b^m c^{n+m} | n, m > 0\}$ **is deterministic?**

3. **Previous pda for** $\{ww^R | w \in \Sigma^+\}$**,**$\Sigma = \{a, b\}$ **is deterministic?**

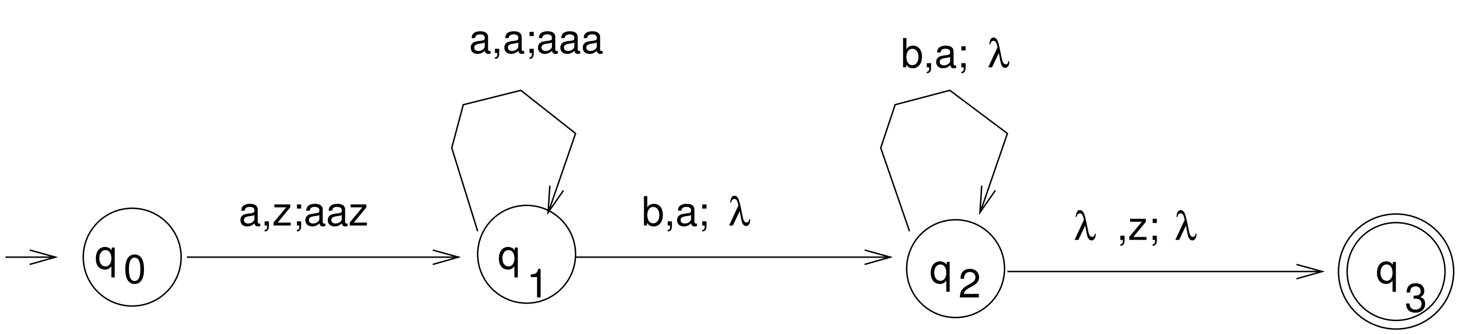
Example: $\mathbf{L}=\{a^n b^m | m > n, m, n > 0\}$,
$\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$



Example: $\mathbf{L}=\{a^n b^{n+m} c^m | n, m > 0\}$,
$\Sigma = \{a, b, c\}$,



Example: $\mathbf{L}=\{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$



Chapter 7.2

Theorem Given NPDA M that accepts by final state, ∃ NPDA M' that accepts by empty stack s.t. L(M)=L(M').

- **Proof (sketch)**

  M=(Q,Σ,Γ,δ,q_0,z,F)

  Construct M'=(Q',Σ,Γ',δ',q_s,z',F')

Theorem Given NPDA M that accepts by empty stack, $\exists$ NPDA M' that accepts by final state.

- **Proof: (sketch)**
  M=(Q,Σ,Γ,δ,q_0,z,F)
  Construct M'=(Q',Σ,Γ',δ',q_s,z',F')

Theorem For any CFL L not containing λ, $\exists$ an NPDA M s.t. L=L(M).

- **Proof (sketch)**
  Given (λ-free) CFL L.
  $\Rightarrow \exists$ CFG G such that L=L(G).
  $\Rightarrow \exists$ G' in GNF, s.t. L(G)=L(G').
  G'=(V,T,S,P). All productions in P are of the form:

Example: Let G'=(V,T,S,P), P=

$$\mathbf{S \to aSA \mid aAA \mid b}$$
$$\mathbf{A \to bBBB}$$
$$\mathbf{B \to b}$$

Theorem Given a NPDA M, $\exists$ a NPDA M' s.t. all transitions have the form $\delta(q_i$,a,A)=$\{c_1, c_2, \ldots c_n\}$ where

$$c_i = (q_j, \lambda)$$
$$\textbf{or } c_i = (q_j, \textbf{BC})$$

Each move either increases or decreases stack contents by a single symbol.

- **Proof (sketch)**

Theorem If L=L(M) for some NPDA M, then L is a CFL.

- **Proof: Given NPDA M.**

  First, construct an equivalent NPDA M that will be easier to work with. Construct M' such that

  1. **accepts if stack is empty**
  2. **each move increases or decreases stack content by a single symbol. (can only push 2 variables or no variables with each transition)**

  M'=(Q,Σ,Γ,δ,q_0,z,F)

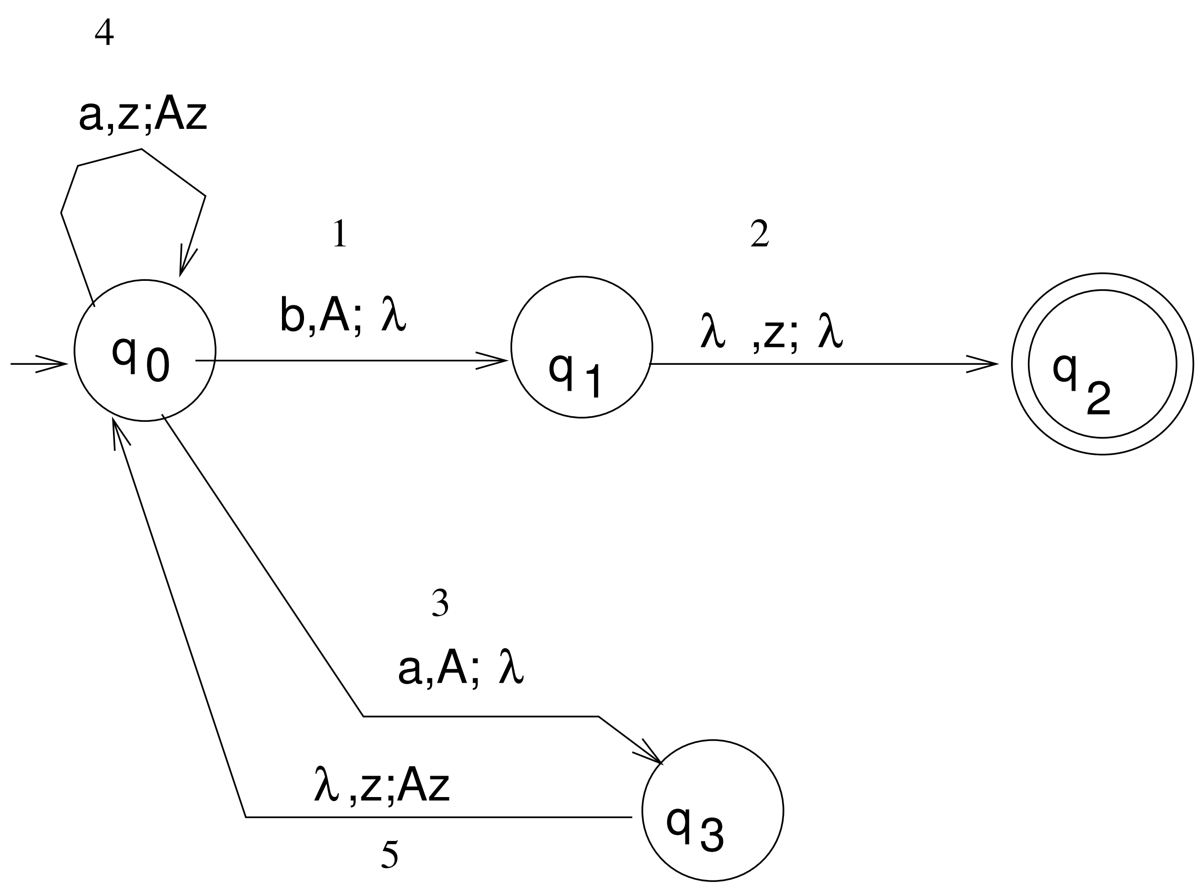
  Construct G=(V,Σ,S,P) where

  V=$\{(q_i c q_j) | q_i, q_j \in Q, c \in \Gamma\}$

  Goal: $(q_0 z q_f)$ **which will be the start symbol in the grammar.**

Example:

L(M)=$\{aa^*b\}$, **M=(Q,**Σ**,**Γ**,**δ**,**q_0**,z,F),**
Q=$\{q_0, q_1, q_2, q_3\}$**,**
Σ**=**$\{a, b\}$**,**Γ**=**$\{A, z\}$**,F=**$\{\}$**.**



Construct the grammar G=(V,T,S,P),

V=$\{(q_0Aq_0),(q_0zq_0),(q_0Aq_1),(q_0zq_1),\ldots\}$

T=Σ

S=(q_0zq_2)

P=

From transition 1 $(q_0Aq_1) \to b$

From transition 2 $(q_1zq_2) \to \lambda$

From transition 3 $(q_0Aq_3) \to a$

From transition 4

$$
\begin{aligned}
(q_0zq_0) \to\ & a(q_0Aq_0)(q_0zq_0)| \\
& a(q_0Aq_1)(q_1zq_0)| \\
& a(q_0Aq_2)(q_2zq_0)| \\
& a(q_0Aq_3)(q_3zq_0) \\
(q_0zq_1) \to\ & a(q_0Aq_0)(q_0zq_1)| \\
& a(q_0Aq_1)(q_1zq_1)| \\
& a(q_0Aq_2)(q_2zq_1)| \\
& a(q_0Aq_3)(q_3zq_1) \\
(q_0zq_2) \to\ & a(q_0Aq_0)(q_0zq_2)| \\
& a(q_0Aq_1)(q_1zq_2)| \\
& a(q_0Aq_2)(q_2zq_2)| \\
& a(q_0Aq_3)(q_3zq_2) \\
(q_0zq_3) \to\ & a(q_0Aq_0)(q_0zq_3)| \\
& a(q_0Aq_1)(q_1zq_3)| \\
& a(q_0Aq_2)(q_2zq_3)| \\
& a(q_0Aq_3)(q_3zq_3)
\end{aligned}
$$

From transition 5

$$
\begin{aligned}
(q_3 z q_0) \rightarrow\ & (q_0 A q_0)(q_0 z q_0) | \\
& (q_0 A q_1)(q_1 z q_0) | \\
& (q_0 A q_2)(q_2 z q_0) | \\
& (q_0 A q_3)(q_3 z q_0) \\
(q_3 z q_1) \rightarrow\ & (q_0 A q_0)(q_0 z q_1) | \\
& (q_0 A q_1)(q_1 z q_1) | \\
& (q_0 A q_2)(q_2 z q_1) | \\
& (q_0 A q_3)(q_3 z q_1) \\
(q_3 z q_2) \rightarrow\ & (q_0 A q_0)(q_0 z q_2) | \\
& (q_0 A q_1)(q_1 z q_2) | \\
& (q_0 A q_2)(q_2 z q_2) | \\
& (q_0 A q_3)(q_3 z q_2) \\
(q_3 z q_3) \rightarrow\ & (q_0 A q_0)(q_0 z q_3) | \\
& (q_0 A q_1)(q_1 z q_3) | \\
& (q_0 A q_2)(q_2 z q_3) | \\
& (q_0 A q_3)(q_3 z q_3)
\end{aligned}
$$

Recognizing aaab in M:

$$
\begin{aligned}
(q_0, aaab, z) &\vdash (q_0, aab, Az) \\
&\vdash (q_3, ab, z) \\
&\vdash (q_0, ab, Az) \\
&\vdash (q_3, b, z) \\
&\vdash (q_0, b, Az) \\
&\vdash (q_1, \lambda, z) \\
&\vdash (q_2, \lambda, \lambda)
\end{aligned}
$$

Derivation of string aaab in G:

$$
\begin{aligned}
(q_0 z q_2) &\Rightarrow a(q_0 A q_3)(q_3 z q_2) \\
&\Rightarrow aa(q_3 z q_2) \\
&\Rightarrow aa(q_0 A q_3)(q_3 z q_2) \\
&\Rightarrow aaa(q_3 z q_2) \\
&\Rightarrow aaa(q_0 A q_1)(q_1 z q_2) \\
&\Rightarrow aaab(q_1 z q_2) \\
&\Rightarrow aaab
\end{aligned}
$$

Chapter 7.3

Definition: A PDA M=(Q,Σ,Γ,δ,q_0,z,F) is *deterministic* **if for every** $q \in$**Q,** $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ **contains at most 1 element**
2. **if** $\delta(q, \lambda, b) \neq \emptyset$ **then** $\delta(q, c, b)=\emptyset$ **for all** $c \in \Sigma$

Definition: L is DCFL iff $\exists$ **DPDA M s.t. L=L(M).**

Examples:

1. **Previous pda for** $\{a^n b^n | n \geq 0\}$ **is deterministic.**
2. **Previous pda for** $\{a^n b^m c^{n+m} | n, m > 0\}$ **is deterministic.**
3. **Previous pda for** $\{ww^R | w \in \Sigma^+\}$**,**$\Sigma = \{a, b\}$ **is nondeterministic.**

Note: There are CFL's that are not deterministic.

L=$\{a^n b^n | n \geq 1\} \cup \{a^n b^{2n} | n \geq 1\}$ **is a CFL and not a DCFL.**

- **Proof:**
  $L = \{a^n b^n : n \geq 1\} \cup \{a^n b^{2n} : n \geq 1\}$
  It is easy to construct a NPDA for $\{a^n b^n : n \geq 1\}$ **and a NPDA for** $\{a^n b^{2n} : n \geq 1\}$**. These two can be joined together by a new start state**

and λ-transitions to create a NPDA for L. Thus, L is CFL.

Now show L is not a DCFL.

Assume that there is a deterministic PDA M such that $L = L(M)$. We will construct a PDA that recognizes a language that is not a CFL and derive a contradiction.

Construct a PDA M' as follows:

1. **Create two copies of M: M_1 and M_2. The same state in M_1 and M_2 are called cousins.**
2. **Remove accept status from accept states in M_1, remove initial status from initial state in M_2. In our new PDA, we will start in M_1 and accept in M_2.**
3. **Outgoing arcs from old accept states in M_1, change to end up in the cousin of its destination in**

M_2. This joins M_1 and M_2 into one PDA. There must be an outgoing arc since you must recognize both $a^n b^n$ and $a^n b^{2n}$. After reading n b's, must accept if no more b's and continue if there are more b's.

4. Modify all transitions that read a b and have their destinations in M_2 to read a c.

This is the construction of our new PDA.

When we read $a^n b^n$ and end up in an old accept state in M_1, then we will transfer to M_2 and read the rest of $a^n b^{2n}$. Only the b's in M_2 have been replaced by c's, so the new machine accepts $a^n b^n c^n$.

The language accepted by our new PDA is $a^n b^n c^n$. But this is not a CFL. Contradiction! Thus there is

no deterministic PDA M such that $L(M) = L$. Q.E.D.