CPS 140 - Mathematical Foundations of CS
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Section: Regular Languages (Ch. 3) (handout)

Regular Expressions

Method to represent strings in a language

$+$	union (or)
$\circ$	concatenation (AND) (can omit)
$*$	star-closure (repeat 0 or more times)

Example:

$(a+b)^* \circ a \circ (a+b)^*$

Example:

$(aa)^*$

Definition Given Σ,

1. $\emptyset$, λ, $a \in \Sigma$ are R.E.
2. If r and s are R.E. then
   - r+s is R.E.
   - rs is R.E.
   - (r) is a R.E.
   - r^* is R.E.
3. r is a R.E. iff it can be derived from (1) with a finite number of applications of (2).

Definition: L(r) = language denoted by R.E. r.

1. $\emptyset$, $\{\lambda\}$, $\{a\}$ are L denoted by a R.E.
2. if r and s are R.E. then
   (a) $L(r+s) = L(r) \cup L(s)$
   (b) $L(rs) = L(r) \circ L(s)$
   (c) $L((r)) = L(r)$
   (d) $L((r)^*) = (L(r)^*)$

Precedence Rules

$*$ highest
$\circ$
$+$

Example:

$ab^* + c =$

Examples:

1. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ has an odd number of a's followed by an even number of b's$\}$.

2. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ has no more than 3 a's and must end in $ab\}$.

3. Regular expression for positive and negative integers

Section 3.2 Equivalence of DFA and R.E.

Theorem Let r be a R.E. Then $\exists$ NFA M s.t. L(M) = L(r).

- Proof:
  $\emptyset$
  $\{\lambda\}$
  $\{a\}$
  Suppose r and s are R.E.
  1. r+s
  2. r$\circ$s
  3. r*

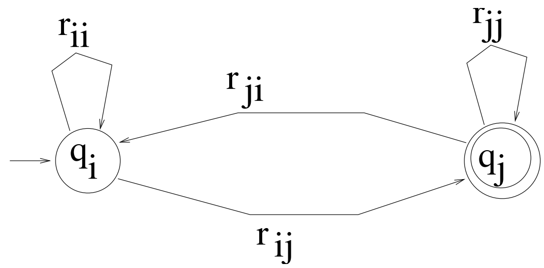
Example

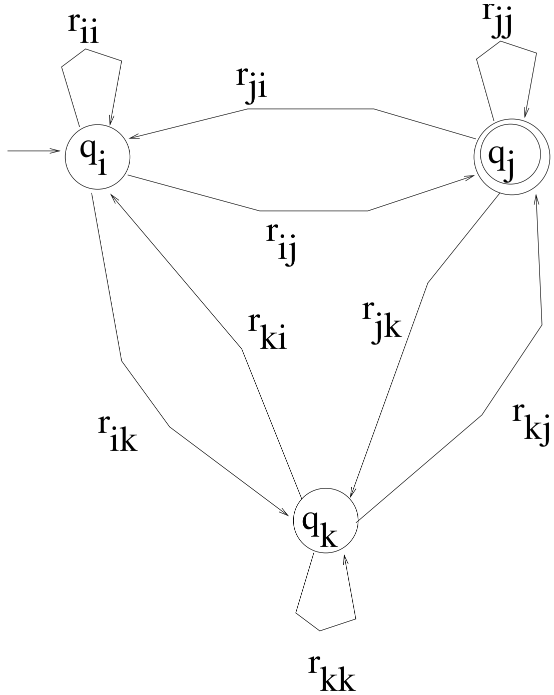
$ab^* + c$

Theorem Let L be regular. Then $\exists$ R.E. r s.t. L=L(r).

Proof Idea: remove states sucessively, generating equivalent generalized transition graphs (GTG) until only two states are left (one initial state and one final state).

- Proof:
  L is regular
  $\Rightarrow \exists$
  1. Assume M has one final state and $q_0 \notin F$
  2. Convert to a generalized transition graph (GTG), all possible edges are present.
  If no edge, label with
  Let r_{ij} stand for label of the edge from q_i to q_j
  3. If the GTG has only two states, then it has the following form:
  In this case the regular expression is:
  $r = (r_{ii}^* r_{ij} r_{jj}^* r_{ji})^* r_{ii}^* r_{ij} r_{jj}^*$
  4. If the GTG has three states then it must have the following form:





In this case, make the following replacements:

REPLACE	WITH
r_{ii}	$r_{ii} + r_{ik}r_{kk}^{*}r_{ki}$
r_{jj}	$r_{jj} + r_{jk}r_{kk}^{*}r_{kj}$
r_{ij}	$r_{ij} + r_{ik}r_{kk}^{*}r_{kj}$
r_{ji}	$r_{ji} + r_{jk}r_{kk}^{*}r_{ki}$

After these replacements, remove state q_k and its edges.

5. If the GTG has four or more states, pick a state q_k to be removed (not initial or final state).

For all $o \neq k, p \neq k$ use the rule

r_{op} replaced with $r_{op} + r_{ok}r_{kk}^{*}r_{kp}$

with different values of o and p.

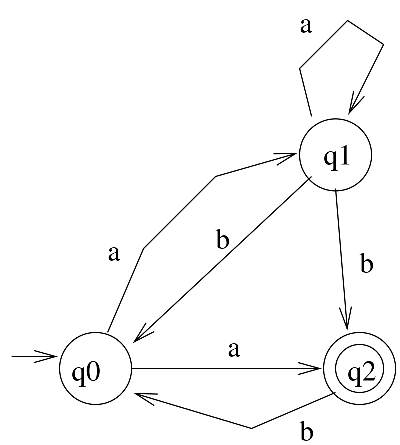
When done, remove q_k and all its edges. Continue eliminating states until only two states are left. Finish with step 3.

6. In each step, simplify the regular expressions r and s with:

$r + r = r$
$s + r^*s =$
$r + \emptyset =$
$r\emptyset =$
$\emptyset^* =$
$r\lambda =$
$(\lambda + r)^* =$
$(\lambda + r)r^* =$

and similar rules.

Example:



Section 3.3

Grammar G=(V,T,S,P)

V	variables (nonterminals)
T	terminals
S	start symbol
P	productions

Right-linear grammar:

all productions of form
$A \rightarrow xB$
$A \rightarrow x$
where A,B $\in$ V, x $\in$ T^*

Left-linear grammar:

all productions of form
$A \rightarrow Bx$
$A \rightarrow x$
where A,B $\in$ V, x $\in$ T^*

Definition:

A regular grammar is a right-linear or left-linear grammar.

Example 1:

G=({S},{a,b},S,P), P=
S $\rightarrow$ abS
S $\rightarrow$ λ
S $\rightarrow$ Sab

Example 2:

G=({S,B},{a,b},S,P), P=
S $\rightarrow$ aB | bS | λ
B $\rightarrow$ aS | bB

Theorem: L is a regular language iff $\exists$ regular grammar G s.t. L=L(G).

Outline of proof:

($\Longleftarrow$) Given a regular grammar G
Construct NFA M
Show L(G)=L(M)
($\Longrightarrow$) Given a regular language
$\exists$ DFA M s.t. L=L(M)
Construct reg. grammar G
Show L(G) = L(M)

Proof of Theorem:

($\Longleftarrow$) Given a regular grammar G
G=(V,T,S,P)
V=$\{V_0, V_1, \ldots, V_y\}$
T=$\{v_o, v_1, \ldots, v_z\}$
S=V_0
Assume G is right-linear
(see book for left-linear case).
Construct NFA M s.t. L(G)=L(M)
If w$\in$L(G), w=$v_1 v_2 \ldots v_k$

M=(V$\cup\{V_f\}$,T,δ,V_0,$\{V_f\}$)
V_0 is the start (initial) state
For each production, $V_i \rightarrow aV_j$,

For each production, $V_i \to a$,

Show L(G)=L(M)

Thus, given R.G. G,

L(G) is regular

($\Longrightarrow$) Given a regular language L

$\exists$ DFA M s.t. L=L(M)

M=(Q,Σ,δ,q_0, F)

Q=$\{q_0, q_1, \ldots, q_n\}$

$\Sigma = \{a_1, a_2, \ldots, a_m\}$

Construct R.G. G s.t. L(G) = L(M)

G=(Q,Σ,q_0,P)

if $\delta(q_i, a_j)$=q_k then

if q_k $\in$F then

Show w $\in$L(M) $\Longleftrightarrow$ w $\in$ L(G)

Thus, L(G)=L(M).

QED.

Example

G=({S,B},{a,b},S,P), P=

S $\to$ aB | bS | λ

B $\to$ aS | bB

Example: