Section: Regular Languages

Regular Expressions

Method to represent strings in a language

$+$ union (or)
$\circ$ concatenation (AND) (can omit)
$*$ star-closure (repeat 0 or more times)

Example:

$(a+b)^* \circ a \circ (a+b)^*$

Example:

$(aa)^*$

Definition Given Σ,

1. $\emptyset$, λ, $a \in \Sigma$ are R.E.
2. If r and s are R.E. then
   - r+s is R.E.
   - rs is R.E.
   - (r) is a R.E.
   - r^* is R.E.
3. r is a R.E. iff it can be derived from (1) with a finite number of applications of (2).

Definition: L(r) = language denoted by R.E. r.

1. $\emptyset$, $\{\lambda\}$, $\{a\}$ **are L denoted by a R.E.**
2. **if r and s are R.E. then**
   (a) $\mathbf{L(r+s) = L(r) \cup L(s)}$
   (b) $\mathbf{L(rs) = L(r) \circ L(s)}$
   (c) $\mathbf{L((r)) = L(r)}$
   (d) $\mathbf{L((r)^*) = (L(r)^*)}$

Precedence Rules

$*$ **highest**

$\circ$

$+$

Example:

$ab^* + c =$

Examples:

1. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ **has an odd number of** a**'s followed by an even number of** b**'s**$\}$.

2. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ **has no more than 3** a**'s and must end in** $ab\}$.

3. **Regular expression for positive and negative integers**

Section 3.2 Equivalence of DFA and R.E.

Theorem Let r be a R.E. Then $\exists$ NFA M s.t. L(M) = L(r).

- **Proof:**

  $\emptyset$

  $\{\lambda\}$

  $\{a\}$

  Suppose r and s are R.E.

  1. **r+s**
  2. **r$\circ$s**
  3. **r***

Example

$ab^* + c$

Theorem Let L be regular. Then $\exists$ R.E. r s.t. L=L(r).

Proof Idea: remove states sucessively until two states left

- **Proof:**

  L is regular

  $\Rightarrow \exists$

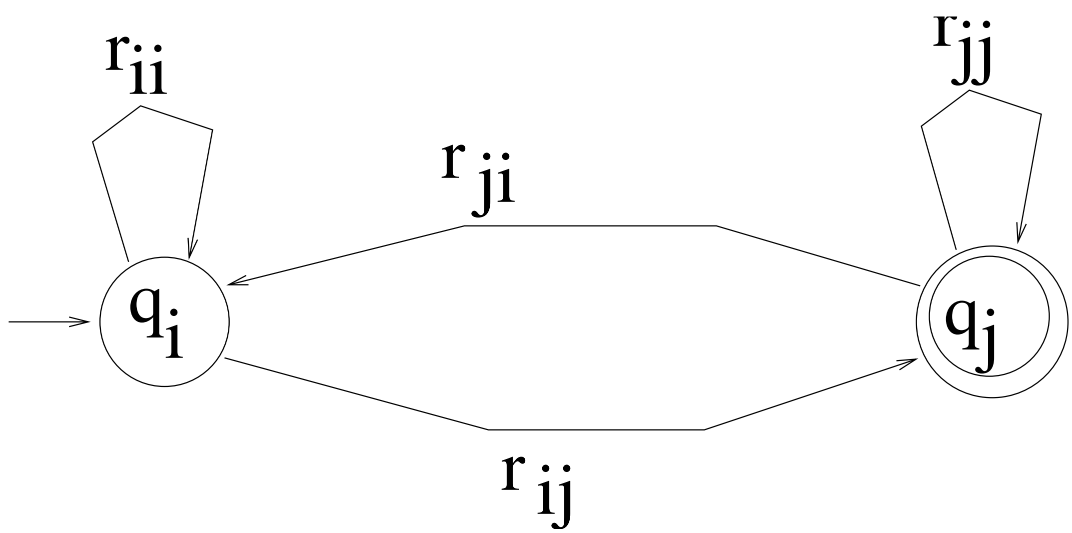
  1. Assume M has one final state and $q_0 \notin F$

  2. Convert to a generalized transition graph (GTG), all possible edges are present.

  If no edge, label with

  Let r_{ij} stand for label of the edge from q_i to q_j

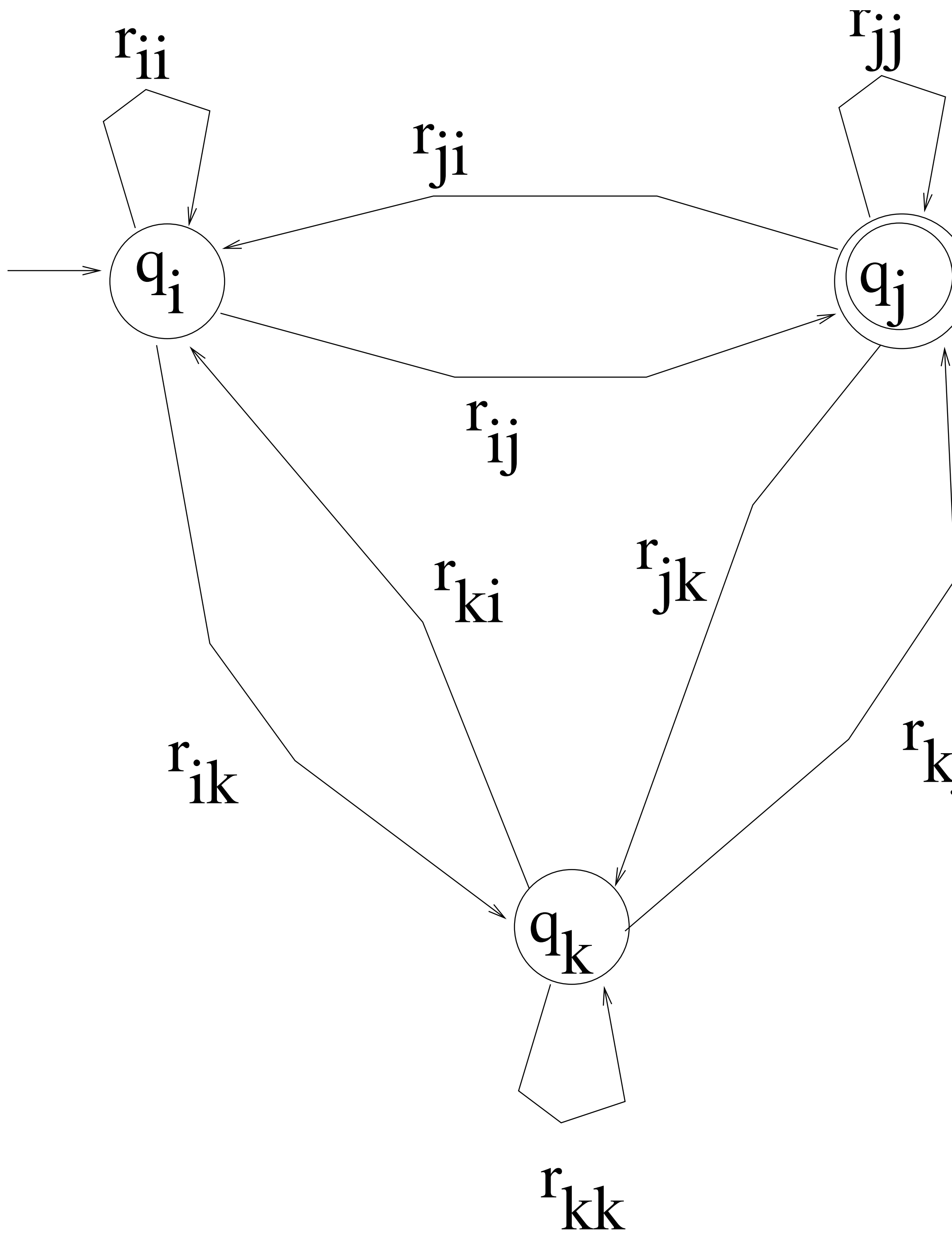
3. If the GTG has only two states, then it has the following form:



In this case the regular expression is:

$r = (r_{ii}^* r_{ij} r_{jj}^* r_{ji})^* r_{ii}^* r_{ij} r_{jj}^*$

4. If the GTG has three states then it must have the following form:



REPLACE	**WITH**
r_{ii}	$r_{ii} + r_{ik} r_{kk}^* r_{ki}$
r_{jj}	$r_{jj} + r_{jk} r_{kk}^* r_{kj}$
r_{ij}	$r_{ij} + r_{ik} r_{kk}^* r_{kj}$
r_{ji}	$r_{ji} + r_{jk} r_{kk}^* r_{ki}$

remove state q_k

5. If the GTG has four or more states, pick a state q_k to be removed (not initial or final state).

For all $o \neq k, p \neq k$ use the rule r_{op} replaced with $r_{op} + r_{ok} r_{kk}^* r_{kp}$ with different values of o and p.

When done, remove q_k and all its edges. Continue eliminating states until only two states are left. Finish with step 3.

6. In each step, simplify the regular expressions r and s with:

$r + r = r$

$s + r^* s =$

$r + \emptyset =$

$r\emptyset =$

$\emptyset^* =$

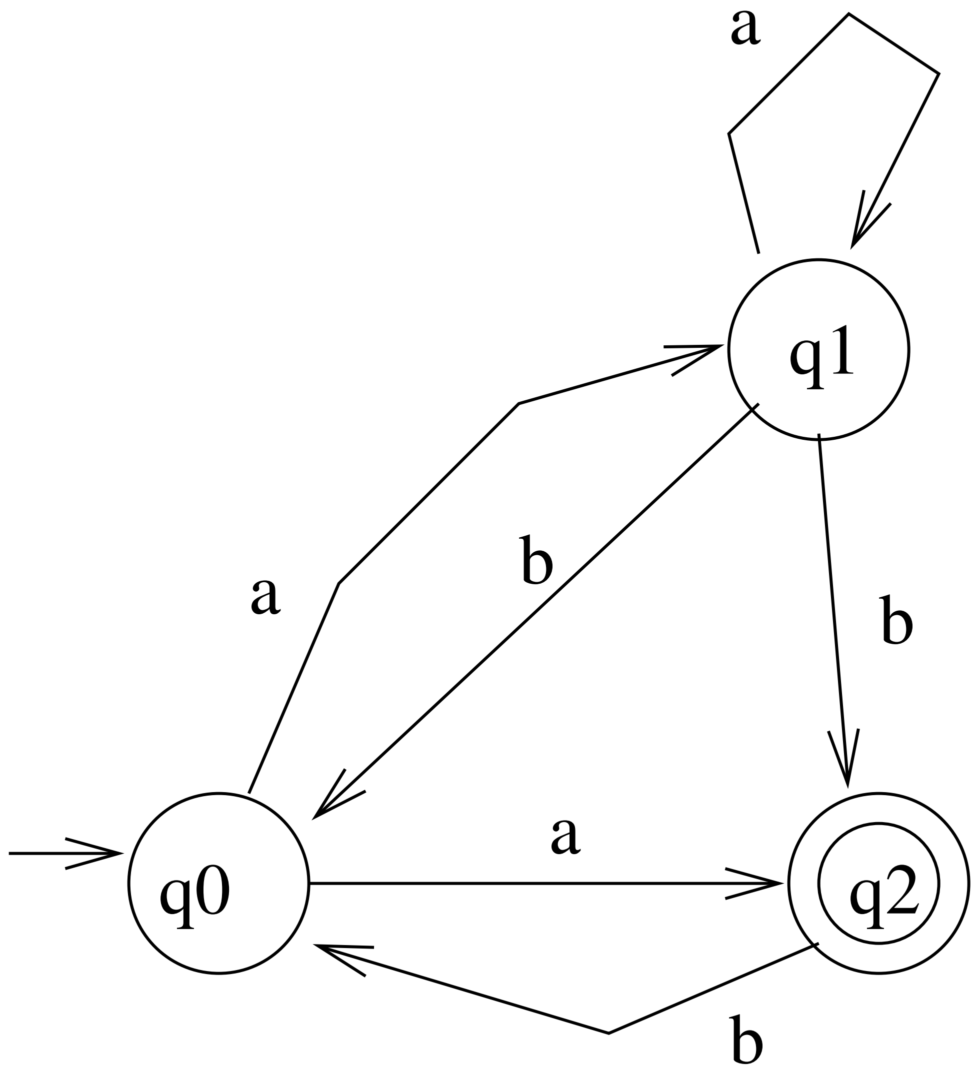
$r\lambda =$

$(\lambda + r)^* =$

$(\lambda + r)r^* =$

and similar rules.

Example:



Grammar G=(V,T,S,P)

V **variables (nonterminals)**
T **terminals**
S **start symbol**
P **productions**

Right-linear grammar:

all productions of form

$$\mathbf{A \to xB}$$
$$\mathbf{A \to x}$$

where $\mathbf{A,B \in V,\ x \in T^*}$

Left-linear grammar:

all productions of form

$$\mathbf{A \to Bx}$$

$$\mathbf{A \to x}$$

where $\mathbf{A,B \in V,\ x \in T^*}$

Definition:

A regular grammar is a right-linear or left-linear grammar.

Example 1:

$$\mathbf{G{=}(\{S\},\{a,b\},S,P),\ P{=}}$$
$$\mathbf{S} \rightarrow \mathbf{abS}$$
$$\mathbf{S} \rightarrow \lambda$$
$$\mathbf{S} \rightarrow \mathbf{Sab}$$

Example 2:

$$\mathbf{G=(\{S,B\},\{a,b\},S,P),\ P=}$$
$$\mathbf{S \rightarrow aB \mid bS \mid} \lambda$$
$$\mathbf{B \rightarrow aS \mid bB}$$

Theorem: L is a regular language iff $\exists$ regular grammar G s.t. L=L(G).

Outline of proof:

($\Longleftarrow$) Given a regular grammar G
Construct NFA M
Show L(G)=L(M)
($\Longrightarrow$) Given a regular language
$\exists$ DFA M s.t. L=L(M)
Construct reg. grammar G
Show L(G) = L(M)

Proof of Theorem:

($\Longleftarrow$) Given a regular grammar G
G=(V,T,S,P)
V=$\{V_0, V_1, \ldots, V_y\}$
T=$\{v_o, v_1, \ldots, v_z\}$
S=V_0
Assume G is right-linear
(see book for left-linear case).
Construct NFA M s.t. L(G)=L(M)
If w$\in$**L(G), w=**$v_1 v_2 \ldots v_k$

M=(V$\cup\{V_f\}$,T,δ,V_0,$\{V_f\}$)

V_0 **is the start (initial) state**

For each production, $V_i \rightarrow aV_j$,

For each production, $V_i \rightarrow a$,

Show L(G)=L(M)

Thus, given R.G. G,

L(G) is regular

(⟹) **Given a regular language L**

∃ DFA M s.t. L=L(M)

M=(Q,Σ**,**δ**,**q_0**, F)**

Q=$\{q_0, q_1, \ldots, q_n\}$

$\Sigma = \{a_1, a_2, \ldots, a_m\}$

Construct R.G. G s.t. L(G) = L(M

G=(Q,Σ**,**q_0**,P)**

if $\delta(q_i, a_j)$**=**q_k **then**

if q_k **∈F then**

Show w ∈L(M) ⟺ **w ∈ L(G)**

Thus, L(G)=L(M).

QED.

Example

$$\mathbf{G=(\{S,B\},\{a,b\},S,P),\ P=}$$
$$\mathbf{S \rightarrow aB \mid bS} \mid \lambda$$
$$\mathbf{B \rightarrow aS \mid bB}$$

Example: