CPS 140 - Mathematical Foundations of CS
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Section: Properties of Regular Languages (Ch. 4) (handout)

**Example**

$L = \{a^n b a^n \mid n > 0\}$

**Closure Properties**

A set is closed over an operation if

$L_1$, $L_2$ $\in$ class
$L_1$ op $L_2$ = $L_3$
$\Rightarrow$ $L_3$ $\in$ class

**Example**

$L_1$={x | x is a positive even integer}

L is closed under

addition?
multiplication?
subtraction?
division?

**Closure of Regular Languages**

**Theorem 4.1** If $L_1$ and $L_2$ are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1 L_2$
$\bar{L}_1$
$L_1^*$

are regular languages.

**Proof**(sketch)

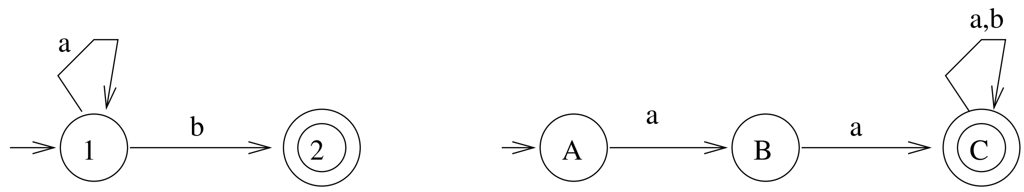
$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure

complementation:
$L_1$ is reg. lang.
$\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$
Construct M' s.t.
final states in M are
nonfinal states in M'
nonfinal states in M are
final states in M'
$\Rightarrow$ closed under complementation

intersection:
$L_1$ and $L_2$ are reg. lang.
$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.
$L_1 = L(M_1)$ and $L_2 = L(M_2)$
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
$Q' = (Q \times P)$
$\delta'$:
$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$
$\Rightarrow$ closed under intersection

**Example:**




**Regular languages are closed under**

| | |
|---|---|
| reversal | $L^R$ |
| difference | $L_1$-$L_2$ |
| right quotient | $L_1/L_2$ |
| homomorphism | h(L) |

**Right quotient**

Def: $L_1/L_2 = \{x | xy \in L_1$ for some $y \in L_2\}$

Example:

$L_1 = \{a^*b^* \cup b^*a^*\}$
$L_2 = \{b^n | n$ is even, $n > 0\}$
$L_1/L_2 =$

**Theorem** If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

**Proof** (sketch)

$\exists$ DFA M=(Q,$\Sigma$,$\delta$,$q_0$,F) s.t. $L_1$ = L(M).

Construct DFA M'=(Q,$\Sigma$,$\delta$,$q_0$,F')

For each state i do
Make i the start state (representing $L_i^{'}$)
if $L_i^{'} \cap L_2 \neq \emptyset$ then
put $q_i$ in F' in M'

QED.

**Homomorphism**

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

h:$\Sigma \rightarrow \Gamma^*$

**Example:**

$\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$

h(a)=11

h(b)=00

h(c)=0

h(bc) =

h(ab$^*$) =

**Questions about regular languages :**

L is a regular language.

- Given L, $\Sigma$, w$\in \Sigma^*$, is w$\in$L?

- Is L empty?

- Is L infinite?

- Does $L_1 = L_2$?

Ch. 4.3 - **Identifying Nonregular Languages**

If a language L is finite, is L regular?

If L is infinite, is L regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} =$
- $L_2 = \{a^n b^n | n > 0\}$

**Prove that** $L_2 = \{a^n b^n | n > 0\}$ **is ?**

- Proof:

**Pumping Lemma:** Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^i z \in L \quad \text{for all } i \geq 0$$

**Meaning:** Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be "pumped" resulting in strings that must be in $L$.

**To Use the Pumping Lemma to prove L is not regular:**

- Proof by Contradiction.

  Assume L is regular.

  $\Rightarrow$ L satisfies the pumping lemma.

  Choose a long string $w$ in L, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).

  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^i z \in$L $\forall$ $i \geq 0$.

  The pumping lemma does not hold. Contradiction!

  $\Rightarrow$ L is not regular. QED.

**Example** L=$\{a^n cb^n | n > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w =$ where $m$ is the constant in the pumping lemma. (Note that $w$ must be choosen such that $|w| \geq m$.)

  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$'s, $y$ contains 1 or more $a$'s, and $z$ contains 0 or more $a$'s concatenated with $cb^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

  It should be true that $xy^i z \in$L for all $i \geq 0$.

**Example** L=$\{a^n b^{n+s} c^s | n, s > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w$=

  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$'s, $y$ contains 1 or more $a$'s, and $z$ contains 0 or more $a$'s concatenated with the rest of the string $b^{m+s}c^s$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

**Example** $\Sigma = \{a, b\}$, L=$\{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w$=

  So the partition is:

**Example** L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof:**

  Assume L is regular. $\Rightarrow$ the pumping lemma holds.

  Choose $w = a^3b^mc^{m-3}$ where $m$ is the constant in the pumping lemma. There are three ways to partition $w$ into three parts, $w = xyz$. 1) y contains only $a$'s 2) y contains only $b$'s and 3) y contains $a$'s and $b$'s

  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide w into three parts s.t. the pumping lemma contraints were true).

  *Case 1:* (y contains only $a$'s). Then $x$ contains 0 to 2 $a$'s, $y$ contains 1 to 3 $a$'s, and $z$ contains 0 to 2 $a$'s concatenated with the rest of the string $b^mc^{m-3}$, such that there are exactly 3 $a$'s. So the partition is:

  $$x = a^k \quad y = a^j \quad z = a^{3-k-j}b^mc^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq 3$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^2z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j-k}b^mc^{m-3}) = a^{3+j}b^mc^{m-3} \notin$ L since $j > 0$, there are too many $a$'s. Contradiction!

  *Case 2:* (y contains only $b$'s) Then $x$ contains 3 $a$'s followed by 0 or more $b$'s, $y$ contains 1 to $m-3$ $b$'s, and $z$ contains 3 to $m-3$ $b$'s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $$x = a^3b^k \quad y = b^j \quad z = b^{m-k-j}c^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq m - 3$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^0z = a^3b^{m-j}c^{m-3} \notin$ L since $j > 0$, there are too few $b$'s. Contradiction!

  *Case 3:* (y contains $a$'s and $b$'s) Then $x$ contains 0 to 2 $a$'s, $y$ contains 1 to 3 $a$'s, and 1 to $m-3$ $b$'s, $z$ contains 3 to $m-1$ $b$'s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $$x = a^{3-k} \quad y = a^kb^j \quad z = b^{m-j}c^{m-3}$$

  where $3 \geq k > 0$, and $m - 3 \geq j > 0$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in$L for all $i \geq 0$.

  $xy^2z = a^3b^ja^kb^mc^{m-3} \notin$ L since $j, k > 0$, there are $b$'s before $a$'s. Contradiction!

  $\Rightarrow$ There is no partition of $w$.

  $\Rightarrow$ L is not regular!. QED.

**To Use Closure Properties** to prove L is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**

  Assume L is regular.

  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

  closure properties $\Rightarrow$ L' is regular.

  Contradiction!

  L is not regular. QED.

**Example** L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof:** (proof by contradiction)

  Assume L is regular.

  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$

  $h(a) = a \quad h(b) = a \quad h(c) = b$

  $h(L) =$

**Example** L=$\{a^n b^m a^m | m \geq 0, n \geq 0\}$

L is not regular.

- **Proof:** (proof by contradiction)
  Assume L is regular.

**Example:** $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.

- **Proof**:
  Assume $L_1$ is regular.
  Goal is to try to construct $\{a^n b^n | n > 0\}$ which we know is not regular.
  Let $L_2 = \{a^*\}$. $L_2$ is regular.
  By closure under right quotient, $L_3 = L_1 \backslash L_2 = \{a^n b^n a^p | 0 \leq p \leq n, n > 0\}$ is regular.
  By closure under intersection, $L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n | n > 0\}$ is regular.
  Contradiction, already proved $L_4$ is not regular!
  Thus, $L_1$ is not regular. QED.