Section: Other Models of Turing Machines

Definition: Two automata are equivalent if they accept the same language.

Turing Machines with Stay Option

Modify δ,

Theorem Class of standard TM's is equivalent to class of TM's with stay option.

Proof:

- **($\Rightarrow$): Given a standard TM M, then there exists a TM M' with stay option such that L(M)=L(M').**

- **($\Leftarrow$): Given a TM M with stay option, construct a standard TM M' such that L(M)=L(M').**

  M=(Q,$\Sigma, \Gamma, \delta, q_0$,**B,F)**

  M'=

  For each transition in M with a move (L or R) put the transition in M'. So, for

  $$\delta(q_i, a) = (q_j, b,\textbf{L or R})$$

  put into δ'

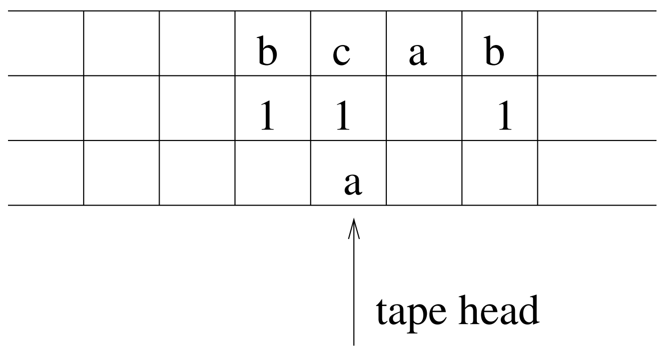
  For each transition in M with S (stay-option), move right and move left. So for

  $$\delta(q_i, a) = (q_j, b,\textbf{S})$$

  L(M)=L(M'). QED.

Definition: **A** *multiple track* **TM divides each cell of the tape into k cells, for some constant k.**

A 3-track TM:



A multiple track TM starts with the input on the first track, all other tracks are blank.

δ**:**

Theorem Class of standard TM's is equivalent to class of TM's with multiple tracks.

Proof: (sketch)

- **($\Rightarrow$): Given standard TM M there exists a TM M' with multiple tracks such that L(M)=L(M').**

- **($\Leftarrow$): Given a TM M with multiple tracks there exists a standard TM M' such that L(M)=L(M').**

Definition: A TM with a semi-infinite tape is a standard TM with a left boundary.

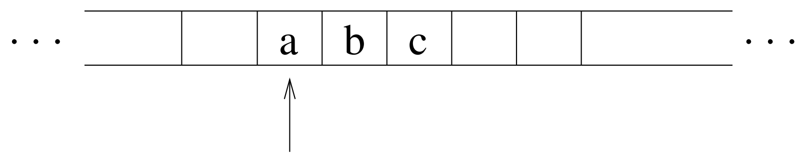
Theorem Class of standard TM's is equivalent to class of TM's with semi-infinite tapes.

Proof: (sketch)

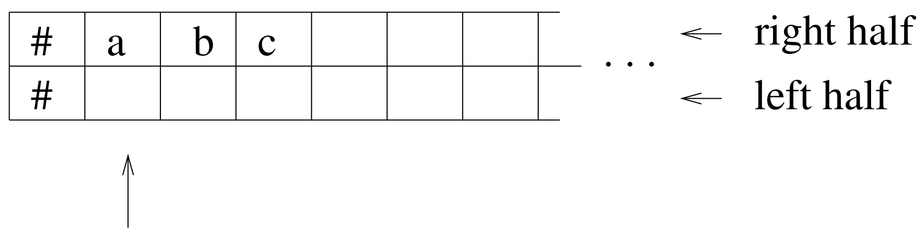
- **($\Rightarrow$): Given standard TM M there exists a TM M' with semi-infinite tape such that L(M)=L(M').**

  Given M, construct a 2-track semi-infinite TM M'

TM M

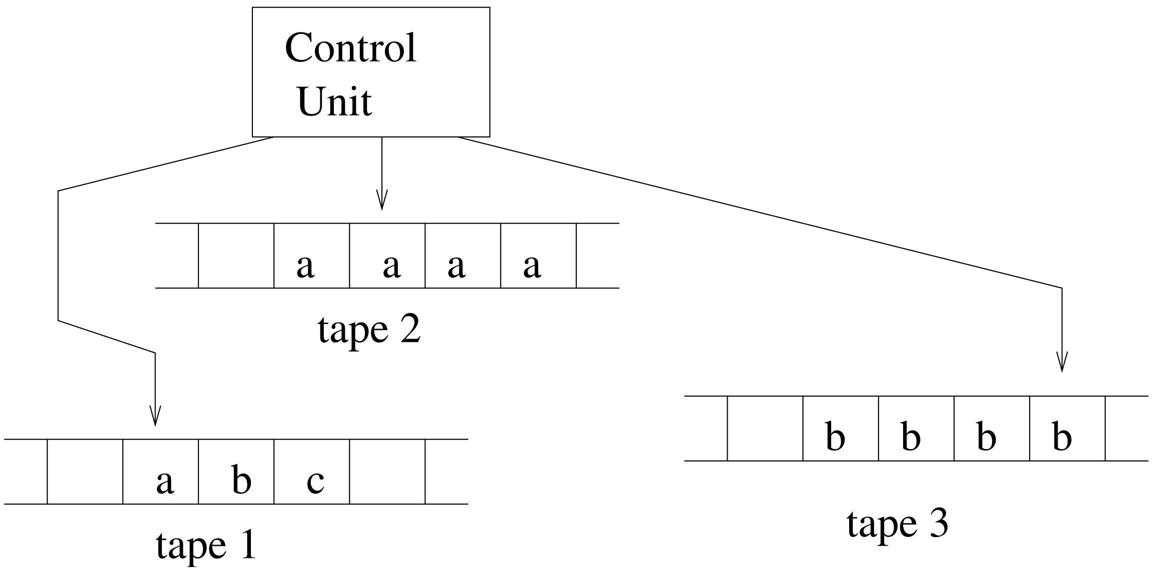


TM M'



- **($\Leftarrow$): Given a TM M with semi-infinite tape there exists a standard TM M' such that L(M)=L(M').**

Definition: An Multitape Turing Machine is a standard TM with multiple (a finite number) read/write tapes.



For an n-tape TM, define δ:

Theorem Class of Multitape TM's is equivalent to class of standard TM's.

Proof: (sketch)

- ($\Leftarrow$): Given standard TM M, construct a multitape TM M' such that L(M)=L(M').

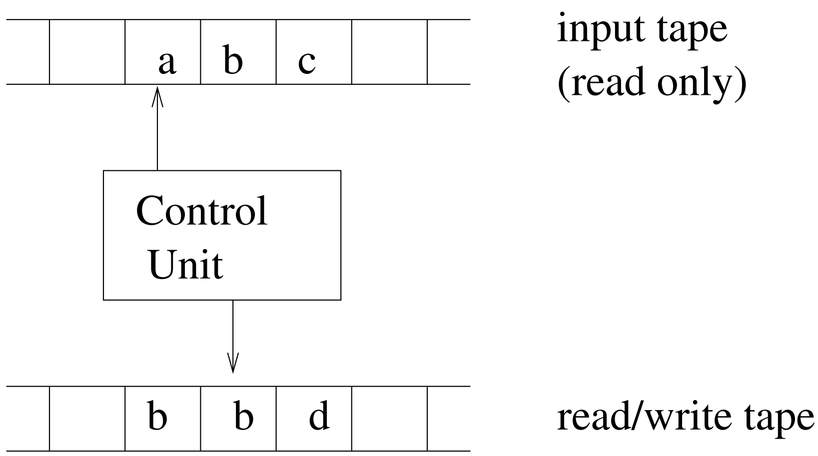
- ($\Rightarrow$): Given n-tape TM M construct a standard TM M' such that L(M)=L(M').

			#	a	b	c				
			#	1						
			#	a	a	a	a			
			#		1					
			#	b	b	b	b			
			#				1			

↑

Definition: An Off-Line Turing Machine is a standard TM with 2 tapes: a read-only input tape and a read/write output tape.

Define δ:



Theorem Class of standard TM's is equivalent to class of Off-line TM's.

Proof: (sketch)

- ($\Rightarrow$): Given standard TM M there exists an off-line TM M' such that L(M)=L(M').

- ($\Leftarrow$): Given an off-line TM M there exists a standard TM M' such that L(M)=L(M').

			#	a	b	c				
			#	1						
			#	b	b	d				
			#		1					

$\uparrow$

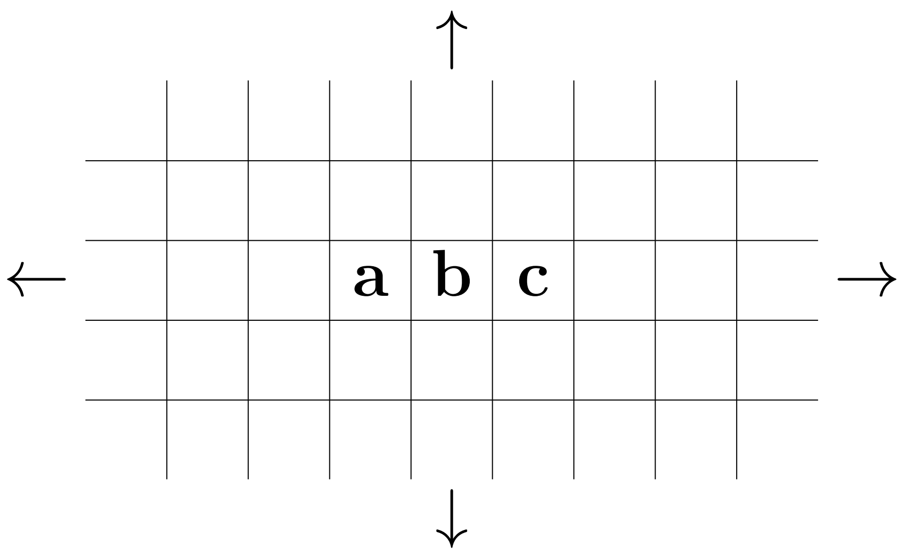
Running Time of Turing Machines

Example:

Given L=$\{a^n b^n c^n | n > 0\}$**. Given w**$\in \Sigma^*$**, is w in L?**

Write a 3-tape TM for this problem.

Definition: An Multidimensional-tape Turing Machine is a standard TM with a multidimensional tape



Define δ:

Theorem Class of standard TM's is equivalent to class of 2-dimensional-tape TM's.

Proof: (sketch)

- **($\Rightarrow$): Given standard TM M, construct a 2-dim-tape TM M' such that L(M)=L(M').**

- **($\Leftarrow$): Given 2-dim tape TM M, construct a standard TM M' such that L(M)=L(M').**

↑

		-1,2	**1,2**	**2,2**				
←	**-2,1**	**-1,1**	**a 1,1**	**b 2,1**	**c 3,1**			
	-2,-1	**-1,-1**	**1,-1**	**2,-1**				

↓

Construct M'

			#	**a**			#	**b**				#	**c**					
			#	**1**	#	**1**	#	**1**	**1**	#	**1**	#	**1**	**1**	**1**	#	**1**	

↑

Definition: A *nondeterministic* Turing machine is a standard TM in which the range of the transition function is a set of possible transitions.

Define δ:

Theorem Class of deterministic TM's is equivalent to class of nondeterministic TM's.

Proof: (sketch)

- **($\Rightarrow$): Given deterministic TM M, construct a nondeterministic TM M' such that L(M)=L(M').**

- **($\Leftarrow$): Given nondeterministic TM M, construct a deterministic TM M' such that L(M)=L(M').**

  Construct M' to be a 2-dim tape TM.

A step consists of making one move for each of the current machines.

For example: Consider the following transition:

$$\delta(q_0, a) = \{(q_1, b, R), (q_2, a, L), (q_1, c, R)\}$$

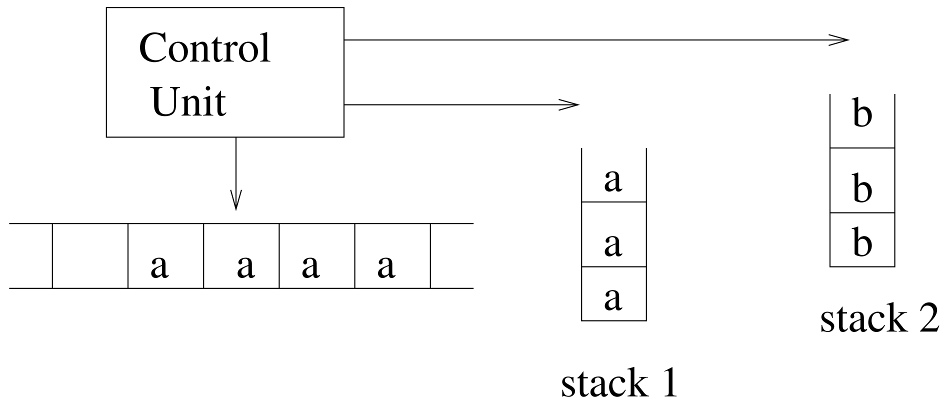
Being in state q_0 with input abc.

#	#	#	#	#
#	**a**	**b**	**c**	#
#	q_0			#
#	#	#	#	#

The one move has three choices, so 2 additional machines are started.

	#	#	#	#	#	#		
	#		**b**	**b**	**c**	#		
	#			q_1		#		
	#		**a**	**b**	**c**	#		
	#	q_2				#		
	#		**c**	**b**	**c**	#		
	#			q_1		#		
	#	#	#	#	#	#		

Definition: A 2-stack NPDA is an NPDA with 2 stacks.



Define δ:

Consider the following languages which could not be accepted by an NPDA.

1. **L=**$\{a^n b^n c^n | n > 0\}$
2. **L=**$\{a^n b^n a^n b^n | n > 0\}$
3. **L=**$\{w \in \Sigma^* |$ **number of** a**'s equals number of** b**'s equals number of** c**'s**$\}$**,** $\Sigma = \{a, b, c\}$

Theorem Class of 2-stack NPDA's is equivalent to class of standard TM's.

Proof: (sketch)

- **($\Rightarrow$): Given 2-stack NPDA, construct a 3-tape TM M' such that L(M)=L(M').**

- **($\Leftarrow$): Given standard TM M, construct a 2-stack NPDA M' such that L(M)=L(M').**

Universal TM - a programmable TM

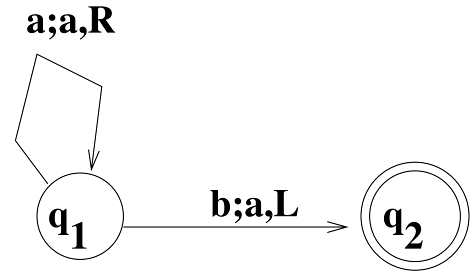
- **Input:**
  - **an encoded TM M**
  - **input string w**
- **Output:**
  - **Simulate M on w**

An encoding of a TM

Let TM M=$\{$**Q,**$\Sigma, \Gamma, \delta, q_1$**,B,F**$\}$

- **Q=**$\{q_1, q_2, \ldots, q_n\}$
  Designate q_1 **as the start state.**
  Designate q_2 **as the only final state.**
  q_n **will be encoded as n 1's**
- **Moves**
  L will be encoded by 1
  R will be encoded by 11
- $\Gamma = \{a_1, a_2, \ldots, a_m\}$
  where a_1 **will always represent the B.**

For example, consider the simple TM:



Γ={B,a,b} which would be encoded as

The TM has 2 transitions,

$$\delta(q_1,\mathbf{a})=(q_1,\mathbf{a},\mathbf{R}), \quad \delta(q_1,\mathbf{b})=(q_2,\mathbf{a},\mathbf{L})$$

which can be represented as 5-tuples:

$$(q_1,\mathbf{a},q_1,\mathbf{a},\mathbf{R}),(q_1,\mathbf{b},q_2,\mathbf{a},\mathbf{L})$$

Thus, the encoding of the TM is:

010110101101101011101101010

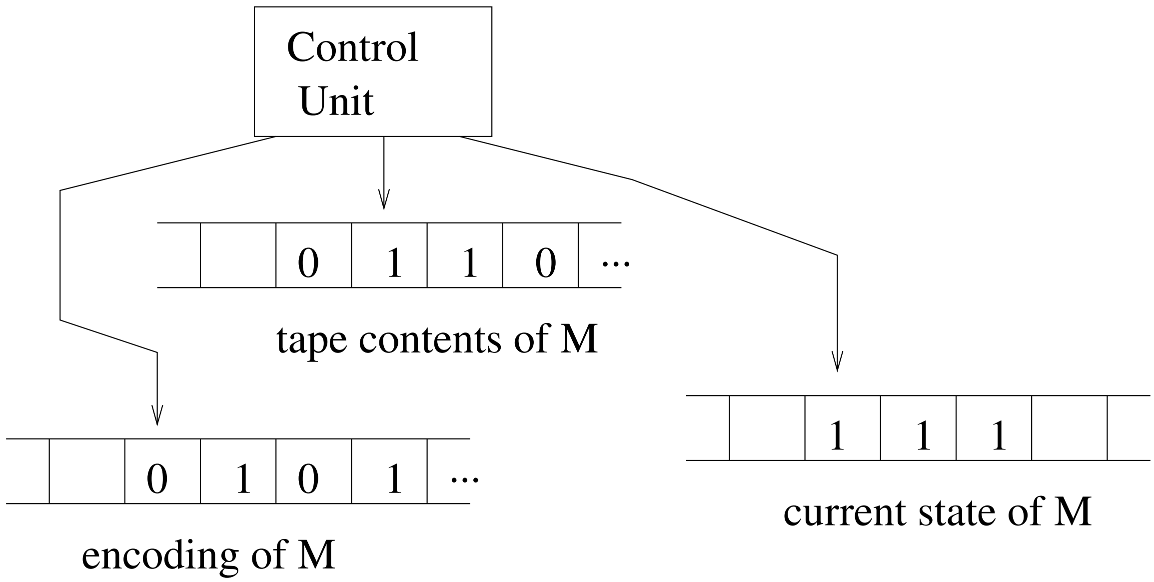
For example, the encoding of the TM above with input string "aba" would be encoded as:

010110101101101011101101101001101110110

Question: Given $w \in \{0, 1\}^+$, is w the encoding of a TM?

Universal TM

The Universal TM (denoted M_U) is a 3-tape TM:



Program for M_U

1. **Start with all input (encoding of TM and string w) on tape 1. Verify that it contains the encoding of a TM.**
2. **Move input w to tape 2**
3. **Initialize tape 3 to 1 (the initial state)**
4. **Repeat (simulate TM M)**
   (a) **consult tape 2 and 3, (suppose current symbol on tape 2 is a and state on tape 3 is p)**
   (b) **lookup the move (transition) on tape 1, (suppose δ(p,a)=(q,b,R).)**
   (c) **apply the move**
      - **write on tape 2 (write b)**
      - **move on tape 2 (move right)**
      - **write new state on tape 3 (write q)**

Observation: Every TM can be encoded as string of 0's and 1's.

Enumeration procedure - process to list all elements of a set in ordered fashion.

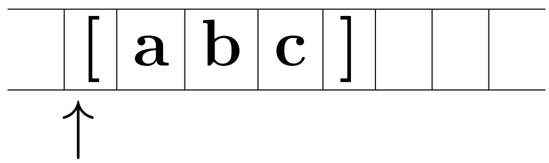
Definition: An infinite set is *countable* **if its elements have 1-1 correspondence with the positive integers.**

Examples:

- **S = { positive odd integers }**
- **S = { real numbers }**
- **S =** $\{w \in \Sigma^+\}$, $\Sigma = \{a, b\}$
- **S = { TM's }**
- **S = {(i,j) | i,j>0, are integers}**

Linear Bounded Automata

We place restrictions on the amount of tape we can use,



Definition: A linear bounded automaton (LBA) is a nondeterministic TM M=(Q,$\Sigma, \Gamma, \delta, q_0$**,B,F) such that** $[,] \in \Sigma$ **and the tape head cannot move out of the confines of []'s. Thus,** $\delta(q_i, [) = (q_j, [, R)$**, and** $\delta(q_i,]) = (q_j,], L)$

Definition: Let M be a LBA.
L(M)=$\{w \in (\Sigma - \{[,]\})^* | q_0[w] \overset{*}{\vdash} [x_1 q_f x_2]\}$

Example: L=$\{a^n b^n c^n | n > 0\}$ **is accepted by some LBA**