Predicate Logic

Today's Menu

Predicate Logic

- Quantifiers: Universal and Existential
- Nesting of Quantifiers
- Applications

Limitations of Propositional Logic

- Suppose we have:

  "All human beings are mortal."

  "Sachin is a human being."

- Does it follow that "Sachin is mortal?"

Cannot be represented using propositional logic.

Need a language that talks about objects, their properties, and their relations.

Predicate Logic

Predicate logic uses the following new features:

- Variables: *x*, *y*, *z* which can be replaced by elements from their domain.
- Predicates: *P*(*x, y*), *M*(*x*) are propositions with variables
- Quantifiers: for all, there exists

Example: P(x, y): x = y + 3.

P(4, 1) is TRUE. ¬P(4, 1) is FALSE.

P(2, 1) is FALSE. ¬P(2, 1) is TRUE.

Note: We talk about the truth value of a propositional function *P*(*x, y*) when we assign values to x and y from their domains, e.g. setting x = 4 and y = 1 to obtain P(4, 1) which is now a proposition.

Quantifiers

Domain of Discourse, U:

The domain of a variable in a propositional function.

Universal Quantification:

P(x) is the proposition: "P(x) is true for all values of x in U."

Universal Quantifier, "For all," symbol: $\forall$

Written as: $\forall x\, P(x)$ which asserts $P(x)$ is true for all x in U.

Existential Quantification:

P(x) is the proposition: "There exists an element x in U such that P(x) is true."

Existential Quantifier, "There exists," symbol: $\exists$

Written as: $\exists\, x\, P(x)$ which asserts $P(x)$ is true for some x in U.

The truth value depends on the choice of U and P(x).

Universal Quantifier $\forall$ (similar to $\wedge$)

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

$\forall x\, P(x)$ Same as $P(x_1) \wedge P(x_2) \ldots \wedge P(x_n) \wedge \ldots$ for all x_i in U

Examples:

1. If P(x) denotes "x is an undergraduate student" and U is {Enorlled Students in COMPSCI 230}, then $\forall x\, P(x)$ is TRUE.
2. If P(x) denotes "x > 0" and U is the integers, then $\forall x\, P(x)$ is FALSE.
3. If P(x) denotes "x > 0" and U is the positive integers, then $\forall x\, P(x)$ is TRUE.
4. If P(x) denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is FALSE.
5. If P(x) denotes "x is mortal" and U represents all human beings, then $\forall x\, P(x)$ is TRUE.

Existential Quantifier $\exists$ (similar to $\vee$)

$\exists x\, P(x)$ is read as "there exists x, P(x)" or "For some x, P(x) or "there exists at least one x such that P(x)"

$\exists x\, P(x)$ Same as $P(x_1) \vee P(x_2) \ldots \vee P(x_n) \vee \ldots$ for all x_i in U

Examples:

1. If P(x) denotes "x is a Duke student" and U is the set of all Enorlled Students in COMPSCI 230, then $\exists x\, P(x)$ is TRUE.
2. If P(x) denotes "x = x + 1" and U is the integers, then $\exists x\, P(x)$ is FALSE.
3. If P(x) denotes "x = x * 2" and U is the integers, then $\exists x\, P(x)$ is TRUE.
4. If P(x) denotes "x is a friend of Mickey mouse" and U is the cartoon characters, then $\exists x\, P(x)$ is TRUE. Namely, Minnie mouse!

5. If P(x) denotes "x is the oldest person in this room" and U is everyone present in the classroom now, then $\exists x\, P(x)$ is TRUE. Namely, your instructor!

Uniqueness Quantifier $\exists!$

How to express the following using quantifiers?

- "There is a unique *x* such that *P*(*x*)."
- "There is one and only one *x* such that *P*(*x*)."

Examples:

1. If P(x) denotes "x + 1 = 0" and U is the integers, then $\exists!x\, P(x)$ is TRUE.
2. But if P(x) denotes "x > 0," then $\exists!x\, P(x)$ is FALSE.

The uniqueness quantifier is **not really needed** as the restriction that there is a unique x such that P(x) can be expressed as:

$$\exists x\,(P(x) \land \forall y\,(P(y) \rightarrow y = x))$$

Thinking about Quantifiers

- When the domain of discourse U is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is TRUE, then $\forall x\, P(x)$ is TRUE.
  - If at a step $P(x)$ is FALSE, then $\forall x\, P(x)$ is FALSE and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is TRUE, then $\exists x\, P(x)$ is TRUE and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is TRUE, then $\exists x\, P(x)$ is FALSE.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Precedence of Quantifiers

Operator	Precedence
$\forall\ \exists$	1
$\neg$	2
$\wedge$ $\vee$	3 4
$\rightarrow$ $\leftrightarrow$	5 6

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

Old Example Revisited

Our Old Example:

- Suppose we have:

  "All human beings are mortal."

  "Sachin is a human being."

- Does it follow that "Sachin is mortal?"

Solution:

- Let $H(x)$: "x is a human being."
- Let $M(x)$: "x is mortal."
- The domain of discourse U is all human beings.
- "All human beings are mortal." translates to $\forall x\, (H(x) \rightarrow M(x))$

  "Sachin is a human being." translates to $H(Sachin)$
- Therefore, for $H(Sachin) \rightarrow M(Sachin)$ to be true it must be the case that $M(Sachin)$. Later we will show this formally.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Take home message: $\neg\forall x P(x) \equiv \exists x \neg P(x)$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

Example: "There is an honest politician."

Let H(x): "x is honest." U consists of all politicians. Then, $\exists x\, H(x)$.

"There does not exist an honest politician." $\neg\exists x\, H(x)$.

$\neg\exists x\, H(x)$ is equivalent to $\forall x\, \neg H(x)$.

However, this statement has a different meaning:

"Not all politicians are honest." How do you express this?

The Lewis Carroll Example

- Premises:
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."

  Conclusion: Can we conclude the following?

  3. "Some fierce creatures do not drink coffee."
- Let $L(x)$: "x is a lion." $F(x)$: "x is fierce." and $C(x)$: "x drinks coffee." Then the above three propositions can be written as:
  1. $\forall x\ (L(x) \rightarrow F(x))$
  2. $\exists x\ (L(x) \wedge \neg C(x))$
  3. $\exists x\ (F(x) \wedge \neg C(x))$
- **Later we'll show how to conclude 3 from 1 and 2.**

Validity and Satisfiability

- An assertion involving predicates and quantifiers is valid if it is true
  - for all domains
  - for every propositional function substituted for the predicates in the assertion.

  Example: $\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$
- An assertion involving predicates is satisfiable if it is true
  - for some domains
  - some propositional functions that can be substituted for the predicates in the assertion.

  Otherwise it is unsatisfiable.

  Example: $\forall x (F(x) \leftrightarrow T(x))$ not valid but satisfiable

  Example: $\forall x (F(x) \wedge \neg F(x))$ unsatisfiable
- The scope of a quantifier is the part of an assertion in which variables are bound by the quantifier.

Nested Quantifiers

Example:	**U when TRUE**	**U when FALSE**
$\forall x \exists y\, (x + y = 0)$	$\mathbb{Z}$	$\mathbb{N}$
$\forall x \forall y\, (xy = yx)$	$\mathbb{Z}$	alphabet
$\exists x \forall y\, (xy = x\,(y + 1))$	$\mathbb{Z}$	$\mathbb{Z}^+$
$\exists x \exists y$ (x has name Mickey and y has name Minnie)	Disney World	COMPSCI 230

Example: Let U be the real numbers, Define P(x,y) : x/y = 1

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$ **Answer:** FALSE
2. $\forall x \exists y P(x,y)$ **Answer:** TRUE
3. $\exists x \forall y\, P(x,y)$ **Answer:** FALSE
4. $\exists x\, \exists\, y\, P(x,y)$ **Answer:** TRUE

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

More Examples

Example 1: "Brothers are siblings."

Solution: $\forall x \, \forall y \, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \, \exists y \, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \, \forall x \, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \, \exists y \, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x \, L(x,x)$