Due Date: March 26, 2013

Problem 1: Let $G = (V, E)$ be an undirected graph with nonnegative edge weights. Suppose we have computed a minimum spanning tree (MST) T of G, and we have also computed shortest paths to all nodes from a particular node $s \in V$. Now, let the weight of each edge be increased by 1.

i) Does the MST change? Give an example where it changes or prove that it cannot change.

ii) Do the shortest paths change? Give an example where they change, or prove they cannot change.

Problem 2: A palindrome is any string that is exactly the same as its reversal, such as A, or NITIN, or MALAYALAM.

i) Design an efficient dynamic programming algorithm to find the length of the longest *subsequence* of a given string, that is also a palindrome. For e.g., the longest palindrome subsequence of BATMANISAFICTIONALCHARACTER is AANISINAA, so given that string as input, your algorithm should return 9.

ii) Any string can be decomposed into a sequence of palindromes. For e.g., the string BUBBASEESABANANA can be broken into palindromes in the following ways (and many others):

   1. BUB + BASEESAB + ANANA
   2. B + U + BB + A + SEES + ABA + NAN + A
   3. B + U + B + B + A + S + E + E + S + A + B + A + N + A + N + A

   Design an efficient dynamic programming algorithm to find the smallest number of palindromes that make up a given input string. For e.g., given the above input string, your algorithm would return 3.

Problem 3: You are given a steel chain made of n links. You can split the chain into 2 pieces, at any location. The cost of splitting a chain, is the total length of the piece, n in this case, regardless of where you split it. Now suppose you want to break the chain into many pieces. The order in which you break the chain can affect the total cost involved in breaking it. For e.g., if you want to break a chain made of 30 links at positions 5 and 17, then making a cut at position 5 first gives a total cost of 30 + 25 = 55, while making a cut at position 17 first gives a total cost of 30 + 17 = 47. Give an efficient dynamic programming algorithm, that given m positions in a chain, finds the minimum cost of cutting the chain into $m + 1$ pieces. (**Hint:** *This problem is similar to the matrix chain multiplication problem.*)

Problem 4: You are given an unlimited supply of dollar notes, of denominations $c_1, c_2, ...c_n$. You want to make change for a customer who is asking for money worth D dollars. In some cases, you might not be able to give the customer the exact amount he asks for. For e.g., if the customer asks for $22, while you only have notes of denominations $5 and $10, then you cannot make change for him.

Give a dynamic programming solution that runs in $O(nD)$ time, which gives a Yes/No answer to the following problem: "Can you make change for D dollars using the n denominations $c_1, c_2, ...c_n$."

Problem 5: A *vertex cover* of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ that includes at least one endpoint of every edge in E. Give a linear-time algorithm for calculating the vertex cover of a tree. Given input as $T = (V, E)$, return the size of the smallest vertex cover of T.

For e.g., in the following tree, possible vertex covers include $\{$A,B,C,D,E,F,G$\}$ and $\{$A,C,D,F$\}$, but not $\{$C,E,F$\}$. The smallest vertex cover has size 3: $\{$B,E,G$\}$.