Online Prediction & Decision Making

CompSci 590.03
Instructor: Ashwin Machanavajjhala

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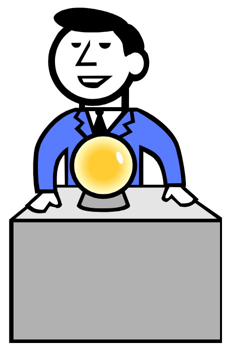
This Class

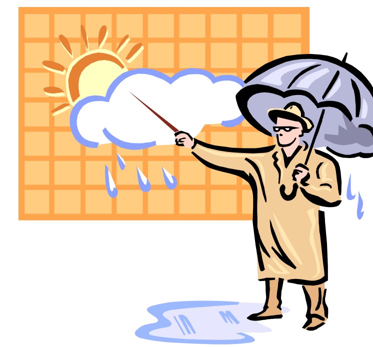
- Weighted Majority Algorithm
  - Multiple experts problem
- Follow the perturbed Leader
  - Online shortest paths
- Multi-armed bandit problems

Multiple Experts Problem

Will it rain today?

Yes **Yes** **Yes** **No**





What is the best prediction based on these experts?

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Multiple Experts Problem

- Suppose we know the best expert (who makes the least error), then we can just return that expert says.
  - This is the best we can hope for.

- We don't know who the best expert is.
  - But we can learn … we know whether it rained or not at the end of the day.

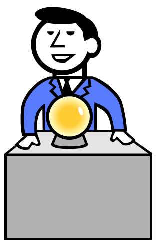
- Regret Minimization : number of mistakes made by our algorithms should be close to the number of mistakes made by the best expert.

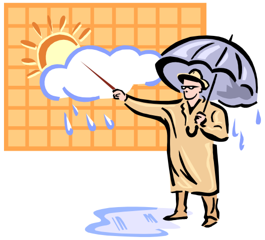
Weighted Majority Algorithm

[Littlestone&Warmuth '94]

"Experts"

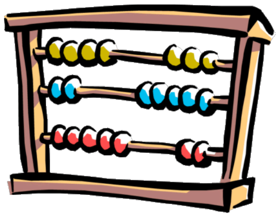
W_1 W_2 W_3 W_4



Y_1 Y_2 Y_3 Y_4

Algorithm



$$Is\ \frac{\sum_i w_i y_i}{\sum_i w_i} > \frac{1}{2}\ ?$$

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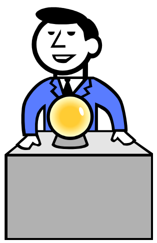
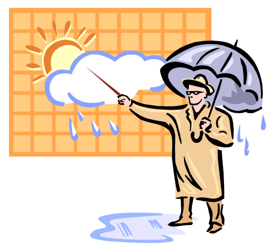
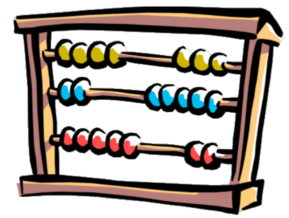
Weighted Majority Algorithm

[Littlestone&Warmuth '94]

"Experts"

1-ε	1-ε	1-ε	1
			
Yes	**Yes**	**Yes**	**No**

Algorithm



$$Is\ \frac{\sum_i w_i y_i}{\sum_i w_i} > \frac{1}{2}\ ?$$

Yes!

Truth

No

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Weighted Majority Algorithm

- Maintain weights (or probability distribution) over experts.

Answering/Prediction:

- Answer using weighted majority, OR
- Randomly pick an expert based on current probability distribution. Use random experts answer.

Update:

- Observe truth.
- Decrease weight (or probability) assigned to the experts who are wrong.

Error Analysis

[Arora, Hazan, Kale '05]

Theorem:

After t steps,

let m(t,j) be the number of errors made by expert j

let m(t) be the number of errors made by algorithm

let n be the number of experts,

$$\forall j, \qquad m(t) \leq \frac{2 \ln n}{\varepsilon} + 2(1+\varepsilon) m(t,j)$$

Error Analysis: Proof

- Let $\varphi(t) = \Sigma w_i$. Then, $\varphi(1) = n$.

- When the algorithm makes a mistake,
  $\varphi(t+1) \leq \varphi(t)\ (1/2 + ½(1-\varepsilon)) = \varphi(t)(1-\varepsilon/2)$
- When the algorithm is correct,
  $\varphi(t+1) \leq \varphi(t)$

- **Therefore,**
  $\boldsymbol{\varphi(t) \leq n(1-\varepsilon/2)^{m(t)}}$

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Error Analysis: Proof

- $\varphi(t) \leq n(1\text{-}\varepsilon/2)^{m(t)}$

- Also, $W_j(t) = (1\text{-}\varepsilon)^{m(t,j)}$

- $\varphi(t) \geq W_j(t) \;\; \Rightarrow n(1\text{-}\varepsilon/2)^{m(t)} \geq (1\text{-}\varepsilon)^{m(t,j)}$

- Hence, $m(t) \geq 2/\varepsilon \ln n + 2(1+\varepsilon)m(t,j)$

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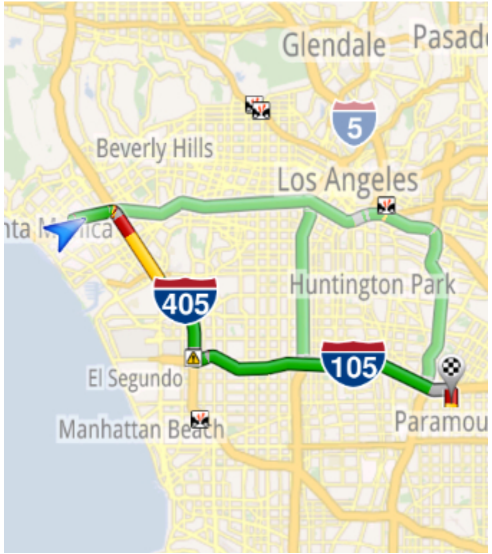
Online Learning

- Mistake bound model
  - Algorithm receives an unlabeled example **x** (like our experts)
  - Algorithm predicts a classification of this example *p* (either -1 or +1)
  - Environment produces the correct answer *y* (either -1 or +1)

- Winnow algorithm
  - Learn a weight function **w** such that *sign*(**w x**) = *p*
  - Same as the Weighted Majority algorithm

Online Shortest Paths Problem

- Input: A directed graph G = (V,E), and a fixed pair of nodes (u,v)
- Each period (time t), we pick a path from u to v, and the length of the path is revealed.
- Cost at time t = length of chosen path.



Online shortest paths

- We could have used weighted majority, where each path is an expert
- But, number of paths (experts) is exponential

Follow the perturbed leader (FPL)

Randomized variant …

Initialization:

- Each expert *j* is assigned a cost *c(j, 0)* = 0

Prediction (time t):

- For each expert j select p(j, t) >= 0 from an exponential distribution ($\mu(x) \sim \varepsilon e^{-\varepsilon x}$)
- Make the same prediction as expert with smallest c(j, t) – p(j, t)

Update:

- If expert j's prediction is correct, c(j, t+1) = c(j, t)
- Else, c(j, t+1) = c(j,t) + 1

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Error Analysis

Theorem:

After t steps,

let m(t,j) be the number of errors made by expert j

let m(t) be the number of errors made by algorithm

let n be the number of experts,

$$E\big(m(t)\big) \leq (1+\varepsilon)m(t,j) + O\left(\frac{1}{\varepsilon}\log n\right)$$

Linear Generalization

- FPL works for more general prediction problems, where
  - The prediction and states are in R^n
  - Total cost of the decisions are $\Sigma\ d_t\ s_t$
  - $\Sigma\ d_t\ s_t$ should be close to $\min_d \Sigma\ d\ s_t$

- Multiple experts:
  - d : 0/1 vector where d[j] = 1 if expert j is picked by the algorithm
  - s : 0/1 vector where s[j] = 0 if jth expert is correct.
  - Total cost is number of mistakes.

Online Shortest Paths

Algorithm:

Initialize all edge costs c(e,0) = 0.

At each time period:

- For each edge, pick p(e, t) from an exponential distribution
- Use the shortest path in the graph with lengths c(e,t) + p(e,t) on each edge.

Online shortest paths

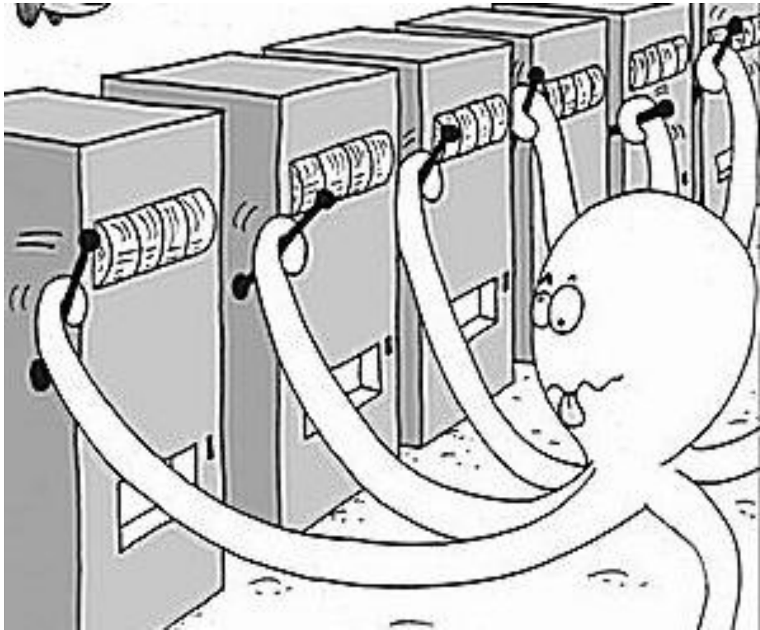
- We could have used weighted majority, where each path is an expert
- But, number of paths (experts) is exponential

- FPL allows solving the problem in polynomial time.

$$E[cost] \leq (1+\varepsilon)(\text{best-time in hindsight}) + \frac{O(mn \log n)}{\varepsilon}$$

Multi-armed Bandit Problem

- A set of actions (or arms)
- Selecting action a in A (or pulling an arm) results in a reward from an unknown probability distribution P(r | a)
- At time=t, agent selects action a_t
- Environment generates reward r_t
- Goal is to maximize $\Sigma_t \, r_t$



Application

- Web advertising
  - What is the best ad/article to show a user?
- Clinical trials
  - Identifying efficient drugs with minimal patient loss/side-effects
- Web search
  - Which result must be ranked at the top?
- …

Regret

- Action value: $Q(a) = E(r \mid a)$ (mean reward)
- Optimal value: $V^* = Q(a^*) = \max_a Q(a)$
- Regret at time t : $E[\, V^* - Q(a_t)]$
- Maximizing cumulative reward is equivalent to minimizing total regret.

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Explore vs Exploit

- Exploit: Make the best decision given the current information
  - Keep pulling the arm with the current best estimate for the reward

- Explore: Gather more information
  - Pull a different arm

- We can estimate the action value Q(a) by Monte Carlo estimation if lever a was pulled $N_t(a)$ times as follows.

$$\widehat{Q_t}(a) = \frac{1}{N_t(a)} \sum r_t 1_{(a_t = a)}$$

Greedy Algorithm

- Start with some initial estimate for Q(a) for all a
- Keep pulling the lever with the estimated action value.

$$a^* = argmax_{a \varepsilon A} \widehat{Q_t}(a)$$

- Continuous Exploitation

- Can get stuck in suboptimal action forever

ε-Greedy

- With probability 1-ε, pull the best level
- With probability ε, choose a random different lever to pull

- Constant Exploration

- Let Δ_a = V* - Q(a). Then total regret at t steps is at least:

$$t \cdot \frac{\varepsilon}{|A|} \sum_{a \varepsilon A} \Delta_a$$

UCB1

[Auer et al 2002]

- Optimism in the face of uncertainty

- Do not dismiss an action unless it is pretty certain that it has a low value.

UCB1

- Estimate an upper confidence bound for each action value

$$P\left[Q(a) > \widehat{Q_t}(a) + \widehat{U_t}(a)\right] < \delta$$

- This depends on the number of times action *a* is selected
  - Small N(a) => Large upper bound (we are not sure Q(a) is small)
  - Large N(a) => small upper bound (estimate of Q(a) is very good)

- Select the action maximizing Upper Confidence Bound (UCB)

$$a_t = argmax_{a \varepsilon A} \widehat{Q_t}(a) + \widehat{U_t}(a)$$

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UCB1

Theorem:
The UCB1 algorithm achieves logarithmic asymptotic total regret

$$\lim_{t \to \infty} R_t \geq 8 \log t \sum_a \Delta_a$$

References

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