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Do quaternions really perform rotations?

We claim that you can use quaternion multiplication to perform a rotation about an arbitrary axis through the origin.

Is it a rotation?

Ken Shoemake sent me this short proof that **p** -> **q p q**$^{-1}$ performs a rotation in 3D:

Quaternion multiplication preserves norms; i.e., N(**pq**) = N(**p**)N(**q**).
This also implies N(**q**$^{-1}$) = N(**q**)$^{-1}$.
Hence N(**q p q**$^{-1}$) = N(**p**).
This proves that **p** -> **q p q**$^{-1}$ is an orthogonal transform in 4D.
Scalar multiplication commutes, so **q** s **q**$^{-1}$ = s.
Thus **p** -> **q p q**$^{-1}$ is an orthogonal transform of the 3D vector of **p**.
There is a continuous path from the identity to every possible action.
This excludes reflections, so **p** -> **q p q**$^{-1}$ is a 3D rotation.

So, is it the right rotation?

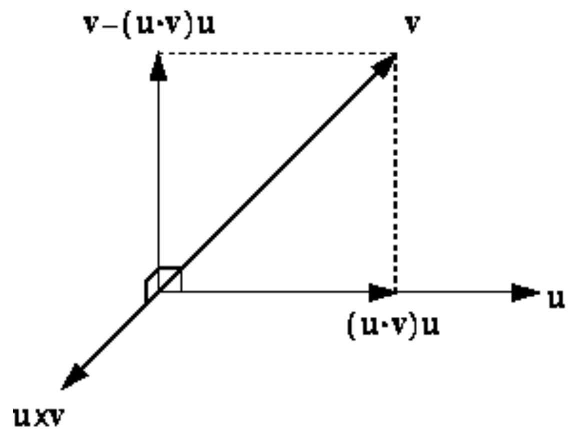
It is easy to see that the axis, represented by the unit quaternion, **u**, is preserved. Since **q** = (cos t/2 + **u** sin t/2) and **u**$^{-1}$ = -**u**, we have:
q = cos t/2 + **u** sin t/2
q^{-1} = cos t/2 + **u**$^{-1}$ sin t/2
u -> **q u q**$^{-1}$
= (cos t/2 + **u** sin t/2) **u** (cost t/2 + **u**$^{-1}$ sin t/2)
= (cos t/2 + **u** sin t/2) (**u** cost t/2 + sin t/2)
= **u** cos^2 t/2 + **u** sin^2 t/2 = **u**

All that remains is to show that the rotation moves through the correct angle. Here is a proof that this operation performs the desired rotation about the axis **u** by the angle .

$$\begin{aligned} \mathbf{p} &= (0, \mathbf{v}) \\ \mathbf{q} &= \left(\cos\frac{\theta}{2}\ ,\ \mathbf{u}\sin\frac{\theta}{2}\right) \\ \mathbf{p}_{\text{rotated}} &= \mathbf{q} * \mathbf{p} * \mathbf{q}^{-1} \end{aligned}$$

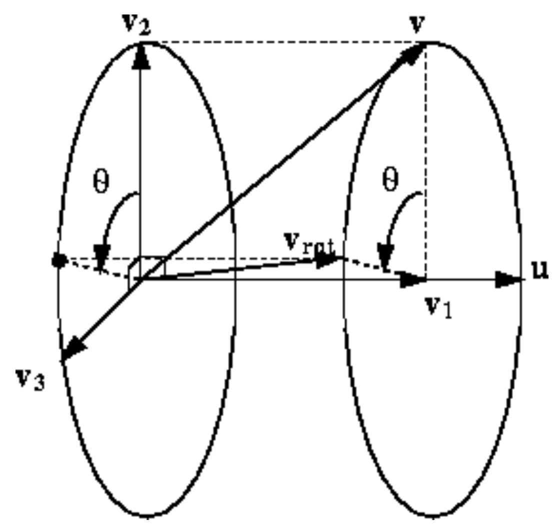
If you would like a hard copy of these notes, a gzipped postscript version can be found in the file quatproof.ps. The LaTeX source can be found in the file quatproof.tex.

Where is the rotated vector?



In this figure, the unit vector **u** is the axis of rotation. The vector **v** will be rotated about **u** by an angle theta to find the vector $\mathbf{v}_{\mathbf{rotated}}$. The vectors **u** and **v** lie in the plane of the page. The vector **u**x**v** points straight up out of the page.

The first step in this proof will be to find a representation for the vector $\mathbf{v}_{\mathbf{rotated}}$ in terms of **u**, **v**, and theta. After that, we will verify that the quaternion operation produces the same vector.



In this second figure, we can see the same vectors as before, but with a few other computed vectors.

- The vector $\mathbf{v_1}$ is the projection of **v** onto **u** and can be computed by $\mathbf{v_1}$ = (**u** . **v**) **u**.
- The vector $\mathbf{v_2}$ is the component of **v** that is orthogonal (perpendicular) to **u** and can be computed by $\mathbf{v_2}$ = **v** - $\mathbf{v_1}$.
- The vector $\mathbf{v_3}$ is orthogonal to **u** and **v** and has the same length as $\mathbf{v_2}$. This vector can be computed by $\mathbf{v_3}$ = **u** x $\mathbf{v_2}$ = **u** x **v** since **u** has unit length and $\mathbf{v_2}$ is orthogonal to **u**.

It is worthwhile to notice at this point that (**u** x **v**) x **u** = $\mathbf{v_3}$ x **u** = $\mathbf{v_2}$ since we will need this fact later in the computation. This is true because **u** is orthogonal to $\mathbf{v_3}$ and $|\mathbf{v_2}| = |\mathbf{v_3}|$.

We can represent the rotated vector $\mathbf{v}_{\mathbf{rotated}}$ in terms of these three vectors by the following equation.

$$\begin{aligned} \mathbf{v}_{\text{rotated}} &= \mathbf{v}_1 + \cos\theta \mathbf{v}_2 + \sin\theta \mathbf{v}_3 \\ \mathbf{v}_{\text{rotated}} &= (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos\theta(\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u}) + \sin\theta(\mathbf{u}\times\mathbf{v}) \end{aligned}$$

Now that we have computed the rotated vector geometrically, we must compute the rotated vector using quaternions and verify that we get the same result.

Verifying the quaternion formula

According to the notes on quaternions, we can compute a rotation by the following quaternion product

$$\begin{aligned} \mathbf{p} &= (0, \mathbf{v}) \\ \mathbf{q} &= \left(\cos\tfrac{\theta}{2}\ ,\ \mathbf{u}\sin\tfrac{\theta}{2}\right) \\ \mathbf{p}_{\text{rotated}} &= \mathbf{q} * \mathbf{p} * \mathbf{q}^{-1} \end{aligned}$$

We will use the vector formula for quaternion multiplication given in the notes.

$$\begin{aligned} \mathbf{q}_1 &= (s_1, \mathbf{v}_1) \\ \mathbf{q}_2 &= (s_2, \mathbf{v}_2) \\ \mathbf{q}_1 * \mathbf{q}_2 &= (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2\ ,\ s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \end{aligned}$$

To verify the formula above: First we substitute our vectors **q** and **p**, noting that $\mathbf{q}^{-1} = \mathbf{q}'$, then we work through the vector products using the above formula.

$$\begin{aligned} \mathbf{p}_{\text{rotated}} &= \mathbf{q} * \mathbf{p} * \mathbf{q}^{-1} \\ &= \left(\cos\tfrac{\theta}{2}\ ,\ \mathbf{u}\sin\tfrac{\theta}{2}\right) * (0\ ,\ \mathbf{v}) * \left(\cos\tfrac{\theta}{2}\ ,\ -\mathbf{u}\sin\tfrac{\theta}{2}\right) \\ &= \left[\left(\cos\tfrac{\theta}{2}\ ,\ \mathbf{u}\sin\tfrac{\theta}{2}\right) * (0\ ,\ \mathbf{v})\right] * \left(\cos\tfrac{\theta}{2}\ ,\ -\mathbf{u}\sin\tfrac{\theta}{2}\right) \\ &= \left(-\sin\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v})\ ,\ \cos\tfrac{\theta}{2}\mathbf{v} + \sin\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\right) * \left(\cos\tfrac{\theta}{2}\ ,\ -\mathbf{u}\sin\tfrac{\theta}{2}\right) \\ &= \Big(-\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v}) + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{v}\cdot\mathbf{u}) - \sin^2\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}, \\ &\quad \sin^2\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \\ &\quad + \cos^2\tfrac{\theta}{2}\mathbf{v} + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v}) \\ &\quad - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{v}\times\mathbf{u}) - \sin^2\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\times\mathbf{u}\Big) \end{aligned}$$

Now we will use some facts about vectors (such as **u** . **v** = **v** . **u** and **u**x**v** = -**v**x**u**) and some trig identities to reduce this equation until it matches the simpler form from the first section. For one thing, we know that (**u**x**v**) . **u** = 0 since (**u**x**v**) is orthogonal to **u**.

$$\begin{aligned} \mathbf{p}_{\text{rotated}} &= \Big(-\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v}) + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{v}\cdot\mathbf{u}) - \sin^2\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}, \\ &\quad \sin^2\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \\ &\quad + \cos^2\tfrac{\theta}{2}\mathbf{v} + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v}) \\ &\quad - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{v}\times\mathbf{u}) - \sin^2\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\times\mathbf{u}\Big) \\ &= \left(0, \sin^2\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos^2\tfrac{\theta}{2}\mathbf{v} + 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v}) - \sin^2\tfrac{\theta}{2}(\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u})\right) \\ &= \left(0, 2\sin^2\tfrac{\theta}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} + (\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2})\mathbf{v} + 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{u}\times\mathbf{v})\right) \\ &= (0, (1-\cos\theta)(\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos\theta\mathbf{v} + \sin\theta(\mathbf{u}\times\mathbf{v})) \\ &= (0, (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos\theta(\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \sin\theta(\mathbf{u}\times\mathbf{v})) \end{aligned}$$

The final line of this equation is the quaternion representation for the point

$$(\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos\theta(\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \sin\theta(\mathbf{u}\times\mathbf{v})$$

which is exactly the point we found using a geometric approach in the first section.

Formulae

The trig identities I used here are

$$\begin{aligned}\cos\theta &= \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\\ \sin\theta &= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\\ \cos\theta &= 1 - 2\sin^2\tfrac{\theta}{2}\end{aligned}$$

Do you have a better proof?

I am very interested in finding a proof of this that is more compact or more geometrically motivated than the one I have presented here. Also, I would like to find a good visualization of what the quaternion product is doing. If you have such a proof or know of one, please *send me mail* and let me know. Thanks a lot!

I am told by Kurt Albrecht from Washington State that almost precisely the same argument for matching rotations and quaternions is presented in Advanced Animation and Rendering Techniques, Theory and Practice by A. Watt and M. Watt

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A shorter proof

Pertti Lounesto of the Helsinki Institute of Technology points out that you can prove this more easily by separating the vector **v** into the components that are parallel and perpendicular to the vector **u**.

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$$

As above, we can also define a third vector which is perpendicular to both the parallel and perpendicular components.

$$\mathbf{v_o} = \mathbf{u}\times\mathbf{v}_{\perp} = \mathbf{u}\times\mathbf{v}$$

with which we can write the rotated point

$$\mathbf{v}_{\text{rotated}} = \mathbf{v}_{||} + \cos\theta\mathbf{v}_{\perp} + \sin\theta\mathbf{v_o}$$

Then we can write the quaternion **p** as

$$\mathbf{p}_{||} = \left(0, \mathbf{v}_{||}\right) \qquad \mathbf{p}_{\perp} = (0, \mathbf{v}_{\perp}) \qquad \mathbf{p} = \mathbf{p}_{||} + \mathbf{p}_{\perp}$$

We can see that the parallel component of **p** commutes with the rotation quaternion **q**

$$\begin{aligned}\mathbf{q} * \mathbf{p}_{||} &= (c,\ s\mathbf{u}) * \left(0, \mathbf{v}_{||}\right)\\ &= \left(0 - s\mathbf{u}\cdot\mathbf{v}_{||}\ ,\ c\mathbf{v}_{||} + 0 + s\mathbf{u}\times\mathbf{v}_{||}\right)\\ &= \left(0 - \mathbf{v}_{||}\cdot s\mathbf{u}\ ,\ 0 + c\mathbf{v}_{||} + \mathbf{v}_{||}\times s\mathbf{u}\right)\\ &= \mathbf{p}_{||} * \mathbf{q}\end{aligned}$$

Then using this information we can verify that $\mathbf{P_{rotated}}$ is the quaternion representing the correctly rotated vector.

$$
\begin{aligned}
\mathbf{p}_{\text{rotated}} &= \mathbf{q} * \mathbf{p} * \mathbf{q}^{-1} \\
&= \mathbf{q} * \left(\mathbf{p}_{||} + \mathbf{p}_{\perp}\right) * \mathbf{q}^{-1} \\
&= \mathbf{q} * \mathbf{p}_{||} * \mathbf{q}^{-1} + \mathbf{q} * \mathbf{p}_{\perp} * \mathbf{q}^{-1} \\
&= \mathbf{q} * \mathbf{q}^{-1} * \mathbf{p}_{||} + \mathbf{q} * \mathbf{p}_{\perp} * \mathbf{q}^{-1} \\
&= \mathbf{p}_{||} + \mathbf{q} * \mathbf{p}_{\perp} * \mathbf{q}^{-1} \\
&= \mathbf{p}_{||} + [(c\ ,\ s\mathbf{u}) * (0\ ,\ \mathbf{v}_{\perp})] * \mathbf{q}^{-1} \\
&= \mathbf{p}_{||} + (-s(\mathbf{u} \cdot \mathbf{v}_{\perp})\ ,\ c\mathbf{v}_{\perp} + s(\mathbf{u} \times \mathbf{v}_{\perp})) * \mathbf{q}^{-1} \\
&= \mathbf{p}_{||} + (0\ ,\ c\mathbf{v}_{\perp} + s\mathbf{v_o}) * (c\ ,\ -s\mathbf{u}) \\
&= \mathbf{p}_{||} + \left(sc(\mathbf{v}_{\perp} \cdot \mathbf{u}) - s^2(\mathbf{v_o} \cdot \mathbf{u}), c^2\mathbf{v}_{\perp} + cs\mathbf{v_o} - cs(\mathbf{v}_{\perp} \times \mathbf{u}) - s^2(\mathbf{v_o} \times \mathbf{u})\right) \\
&= \mathbf{p}_{||} + \left(0, c^2\mathbf{v}_{\perp} + cs\mathbf{v_o} + cs\mathbf{v_o} - s^2\mathbf{v}_{\perp}\right) \\
&= \mathbf{p}_{||} + \left(0, (\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2})\mathbf{v}_{\perp} + 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\mathbf{v_o}\right) \\
&= \mathbf{p}_{||} + (0, \cos\theta\mathbf{v}_{\perp} + \sin\theta\mathbf{v_o}) \\
&= \mathbf{p}_{||} + \cos\theta\mathbf{p}_{\perp} + \sin\theta\mathbf{p_o} \\
&= \mathbf{p}_{\text{rotated}}
\end{aligned}
$$

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