Understanding Quaternions:

Rotations, Reflections, and Perspective Projections

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The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which *__can only be compared for its importance with the invention of triple coordinates by Descartes__*. *The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science. -- Clerk Maxwell, 1869.*

Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell. – Lord Kelvin, 1892

Motivation

Classical Applications of Quaternions in Computer Graphics

Provide Compact Representations for Rotations and Reflections of Vectors in 3-Dimensions

Avoid Distortions due to Floating Point Computations during Rotations

Enable Key Frame Animation by Spherical Linear Interpolation

Compact Representation for Rotations of Vectors in 3-Dimensions

- 3×3 Matrices -- 9 Entries

- Unit Quaternions -- 4 Coefficients

Avoids Distortions due to Floating Point Computations

Problem

- After several matrix multiplications, rotation matrices may no longer be orthogonal due to floating point inaccuracies.

- Non-Orthogonal matrices are difficult to renormalize.
  -- Leads to distortions in lengths and angles during rotation.

Solution

- Quaternions are easily renormalized.
  -- $q \to \dfrac{q}{\|q\|}$ avoids distortions during rotation.

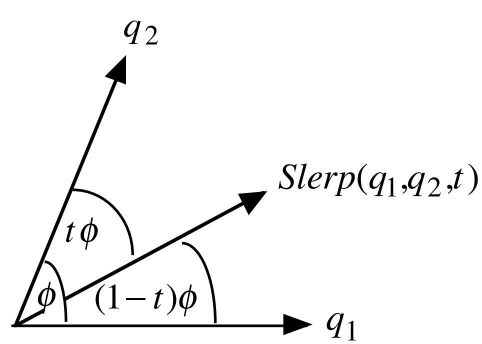
Key Frame Animation

- Linear Interpolation between two rotation matrices R_1 and R_2 (key frames) fails to generate another rotation matrix.

  -- $Lerp(R_1, R_2, t) = (1-t)R_1 + tR_2$ -- not necessarily orthogonal matrices.

- Spherical Linear Interpolation between two unit quaternions always generates a unit quaternion.

  -- $Slerp(q_1, q_2, t) = \frac{\sin((1-t)\phi)}{\sin(\phi)} q_1 + \frac{\sin(t\phi)}{\sin(\phi)} q_2$ -- always a unit quaternion.



Additional Applications of Quaternions in Geometric Modeling

Practical methods for tubing and texturing smooth curves and surfaces using optimal orthonormal frames [Hanson, 2006].

Better ways to visualize streamlines [Hanson, 2006].

Effective techniques for generating and analyzing 3–dimensional Pythagorean hodograph curves [Farouki, 2008].

Novel constructions of curves and surface patches on spheres [Krasauskas, 2011].

Efficient conformal transformations on triangular meshes [Schroder et al, 2011].

Goals and Motivation

- To provide a *geometric interpretation for quaternions*, appropriate for contemporary Computer Graphics.

- To present better ways to *visualize quaternions*, and the effect of quaternion multiplication on points and vectors in 3-dimensions.

- To develop *simple, intuitive proofs of the sandwiching formulas* for rotation and reflection.

- To show how to apply *sandwiching to compute perspective projections* (NEW).

Prerequisites

Complex Numbers

- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- $z \mapsto e^{i\theta} z$ rotates z by the angle θ in the complex plane

Vector Geometry

- $u \cdot v = 0 \Leftrightarrow u \perp v$
- $u \times v \perp u, v$

Models for Visualizing 4–Dimensions

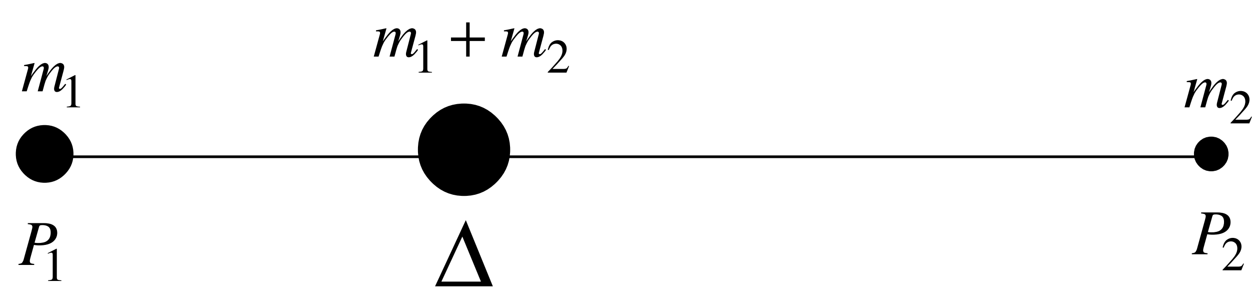
Mathematical Models for 4-Dimensions

- Mass-Points

- Vectors in 4-Dimensions

- Pairs of Mutually Orthogonal Planes

Mass Points and Archimedes' Law of the Lever



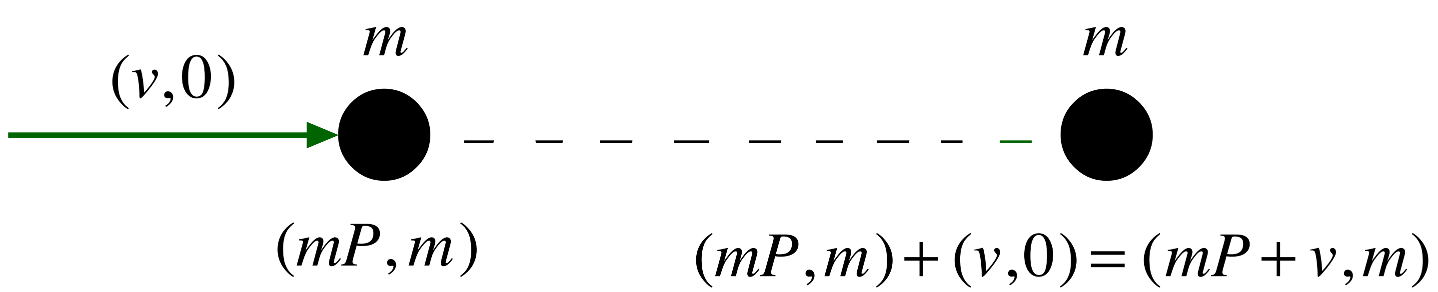
$$(m_1 P_1, m_1) + (m_2 P_2, m_2) = (m_1 P_1 + m_2 P_2, m_1 + m_2)$$

$$d_1 = dist\left(\frac{m_1 P_1 + m_2 P_2}{m_1 + m_2}, P_1\right) = \frac{m_2 \,|\, P_2 - P_1 \,|}{m_1 + m_2}$$

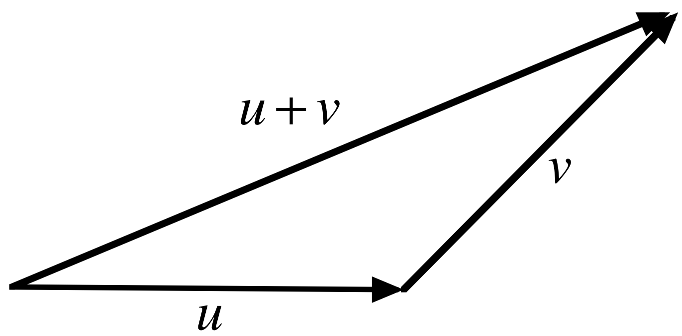
$$\Rightarrow \quad m_1 d_1 = m_2 d_2$$

$$d_2 = dist\left(\frac{m_1 P_1 + m_2 P_2}{m_1 + m_2}, P_2\right) = \frac{m_1 \,|\, P_1 - P_2 \,|}{m_1 + m_2}$$

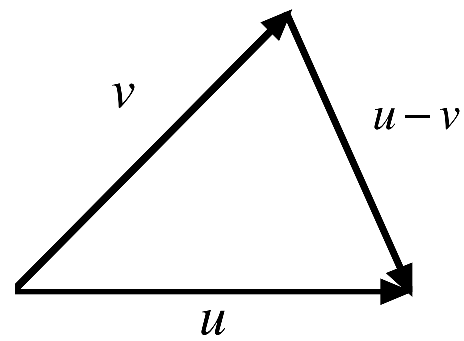
Mass Points and Vectors



Addition and Subtraction for Vectors



Addition



Subtraction

Quaternions

Old Definition

- $q = a + bi + cj + dk = a + \mathbf{v}$
- Sum of a scalar and a vector

New Definition

- $q = aO + bi + cj + dk = aO + \mathbf{v}$
- Sum of a mass-point and a vector = a mass-point
- $O = (0,0,0,1) =$ origin $\leftrightarrow$ identity for quaternion multiplication

Quaternion Multiplication

Notation (Mass-Points)

- $q = aO + bi + cj + dk$
- $O =$ identity for multiplication

Multiplication (Basis Vectors)

- $i^2 = j^2 = k^2 = -O \qquad O^2 = O$
- $ij = k \qquad jk = i \qquad ki = j$
- $ji = -k \qquad kj = -i \qquad ik = -j$

Multiplication (Arbitrary Quaternion)

- $(aO + \boldsymbol{v})(\alpha O + \boldsymbol{w}) = (a\alpha - \boldsymbol{v} \cdot \boldsymbol{w})O + (\alpha \boldsymbol{v} + a\boldsymbol{w} + \boldsymbol{v} \times \boldsymbol{w})$
- $vw = -(\boldsymbol{v} \cdot \boldsymbol{w})O + \boldsymbol{v} \times \boldsymbol{w}$

Properties of Quaternion Multiplication

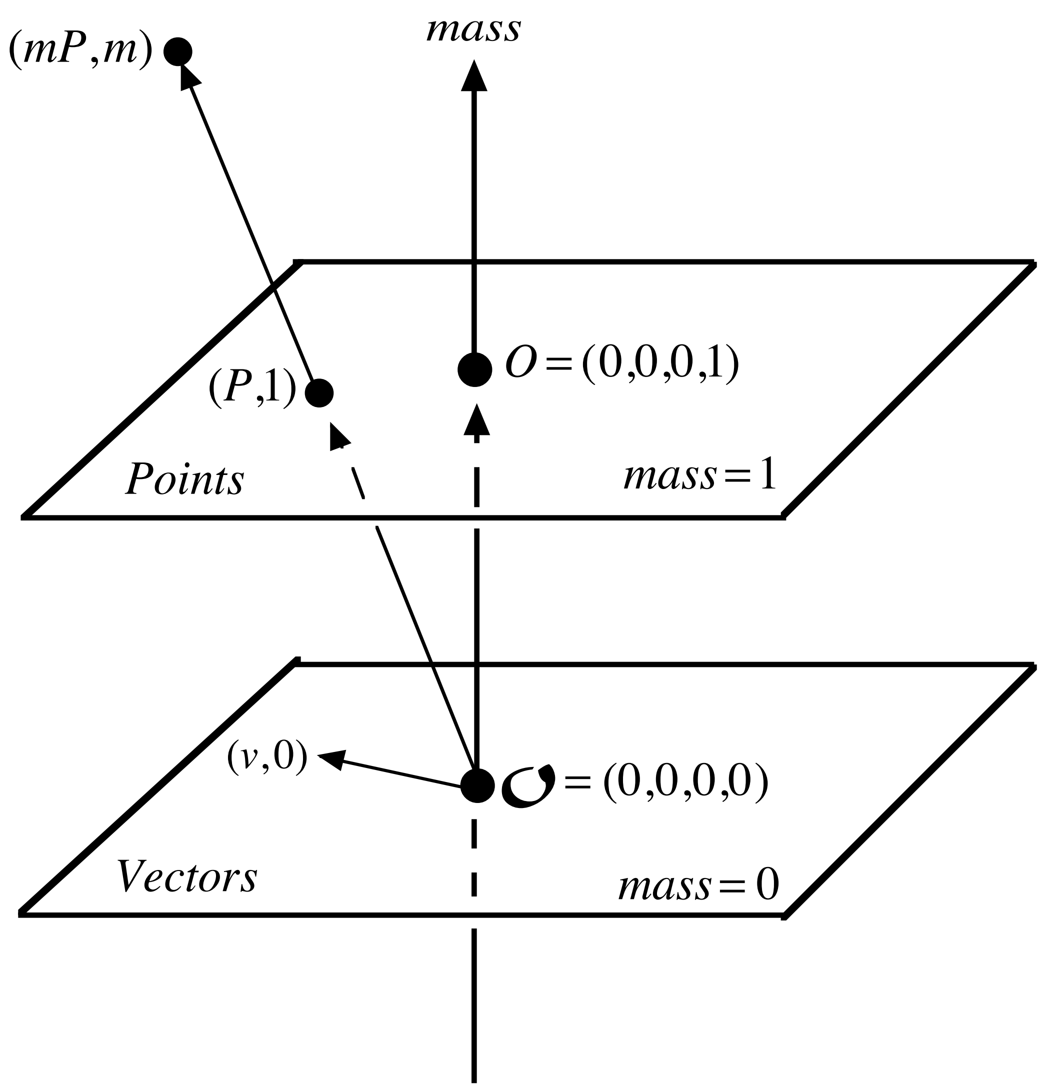
- Associative

- Not Commutative

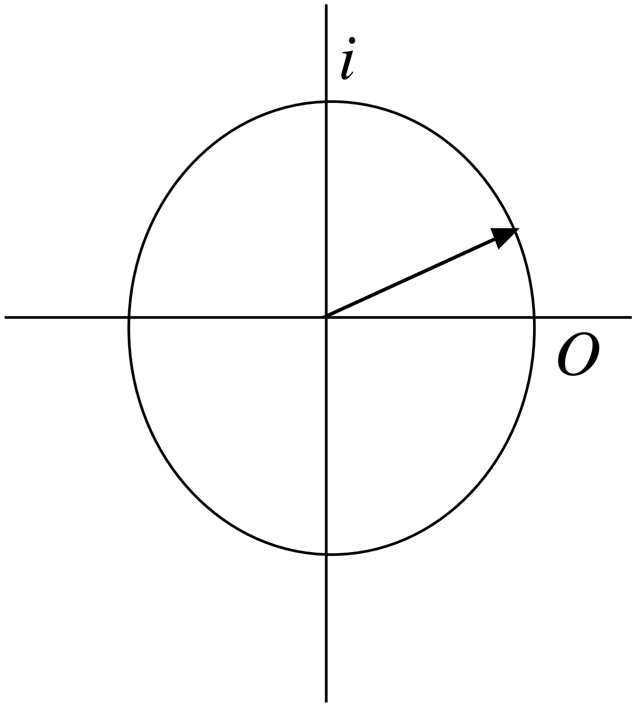
- Distributes Through Addition

- Identity and Inverses

The 4–Dimensional Vector Space of Quaternions

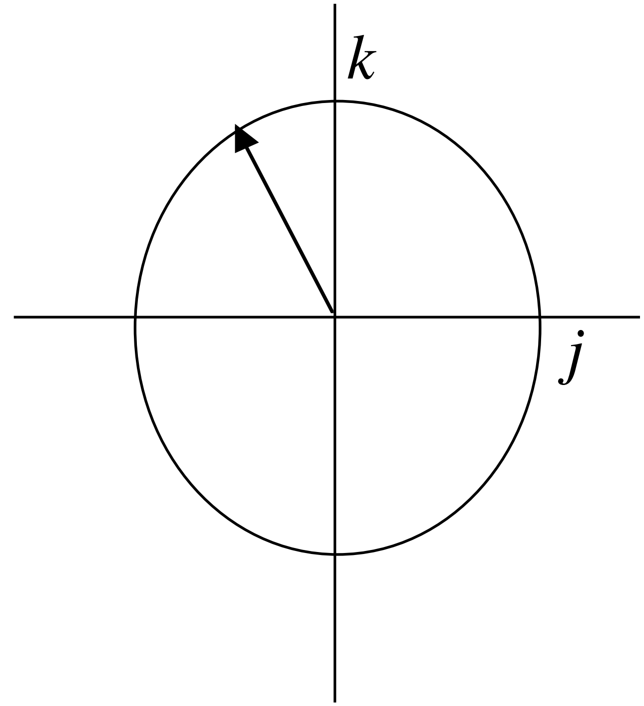


Pairs of Complementary Orthogonal Planes



Complex Plane

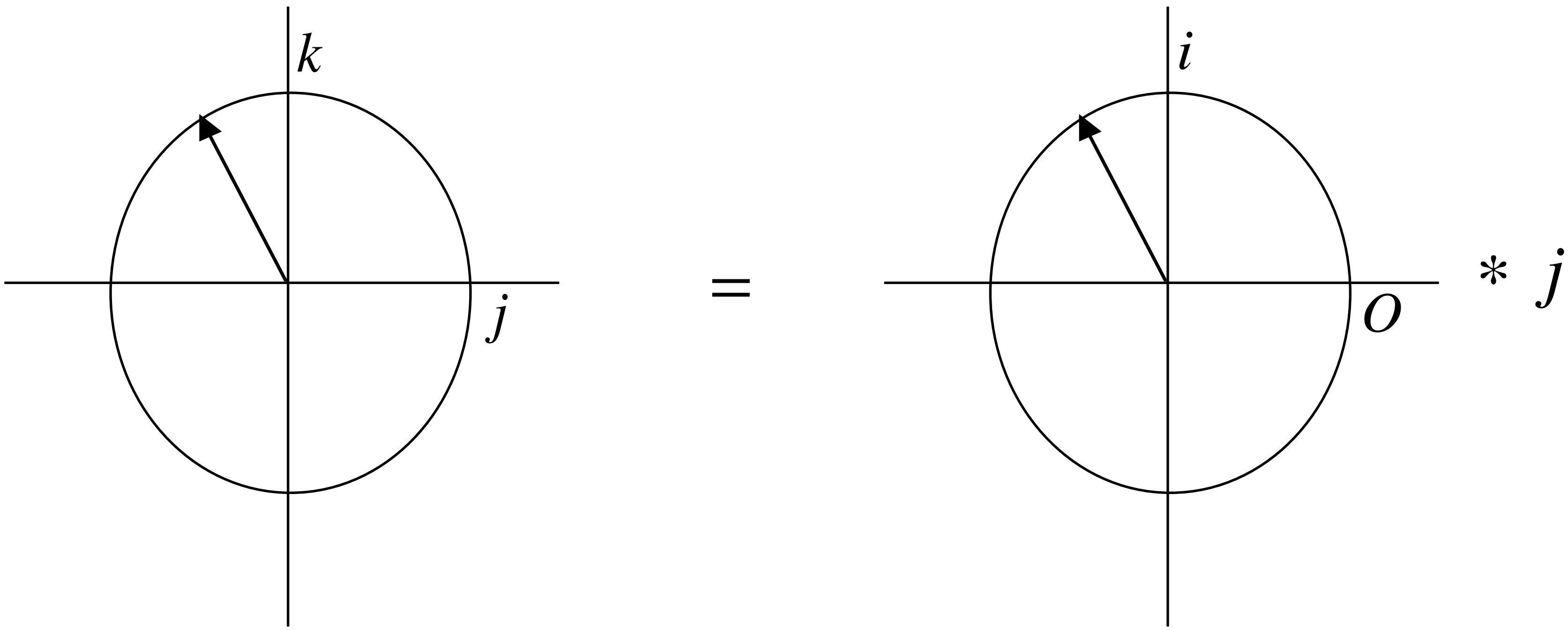
$i^2 = -O\,,\;\; O^2 = O$



Orthogonal Plane

$j, k \perp O, i$

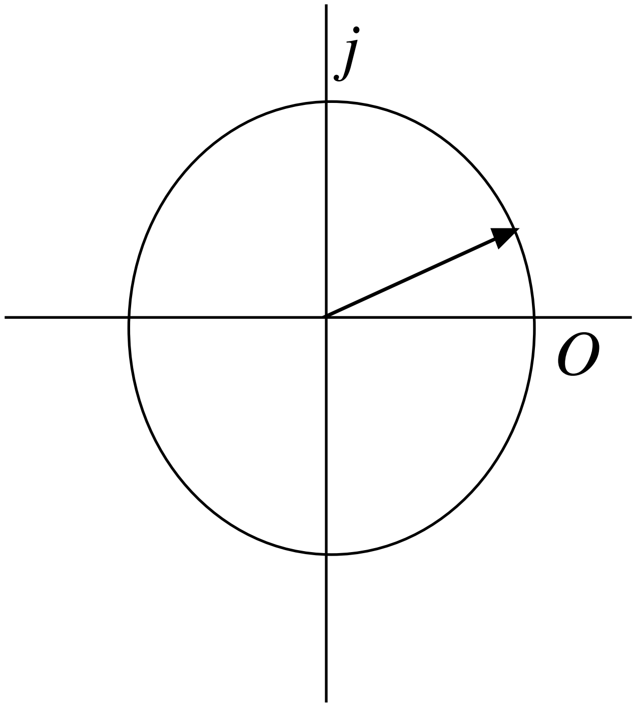
<u>**Orthogonal Plane**</u>



Orthogonal Plane — *Complex Plane * j*

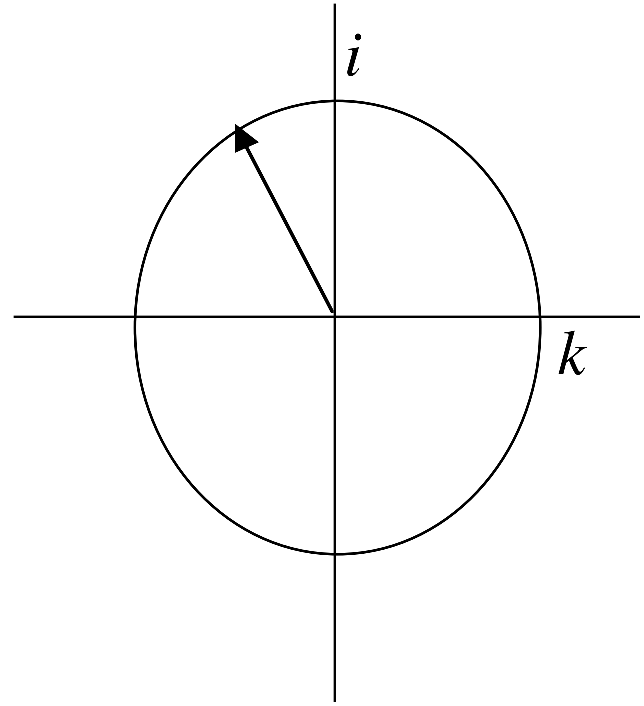
$j, k \perp O, i$

<u>Pairs of Complementary Orthogonal Planes</u>



Complex Plane

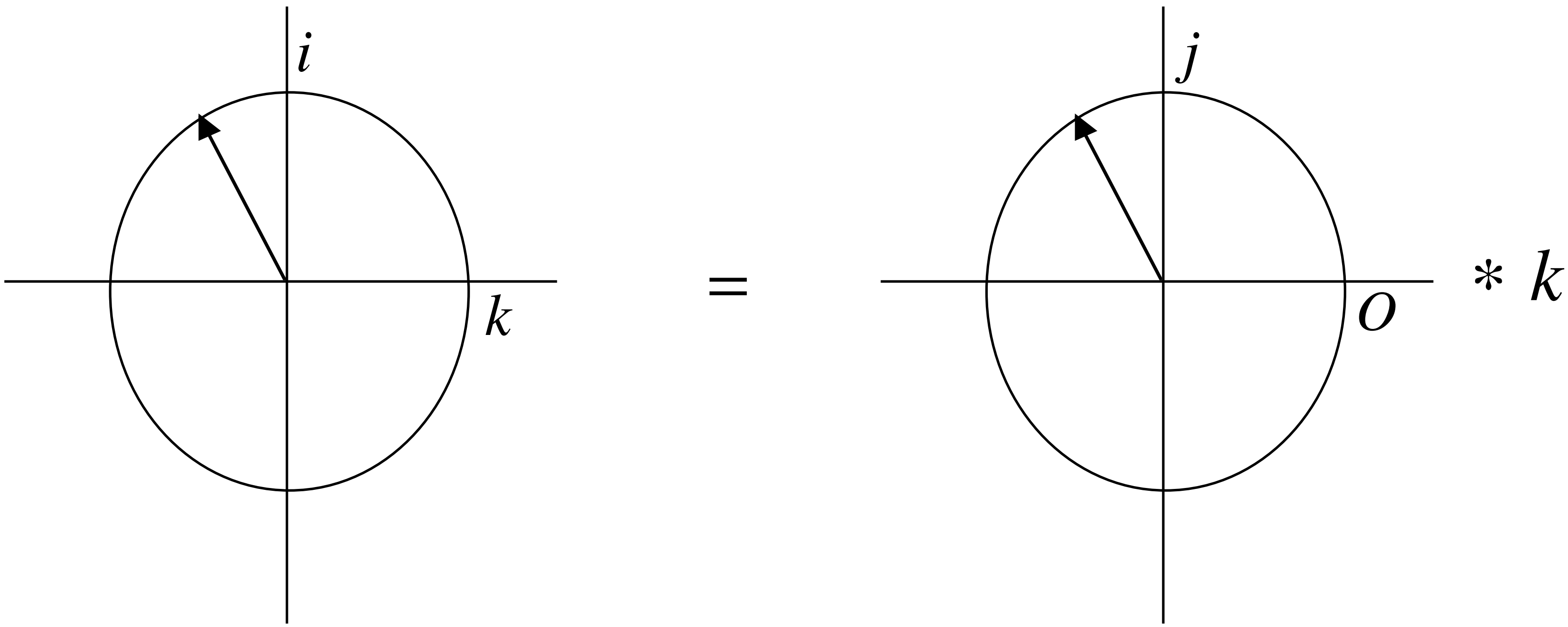
$j^2 = -O\,,\;\; O^2 = O$



Orthogonal Plane

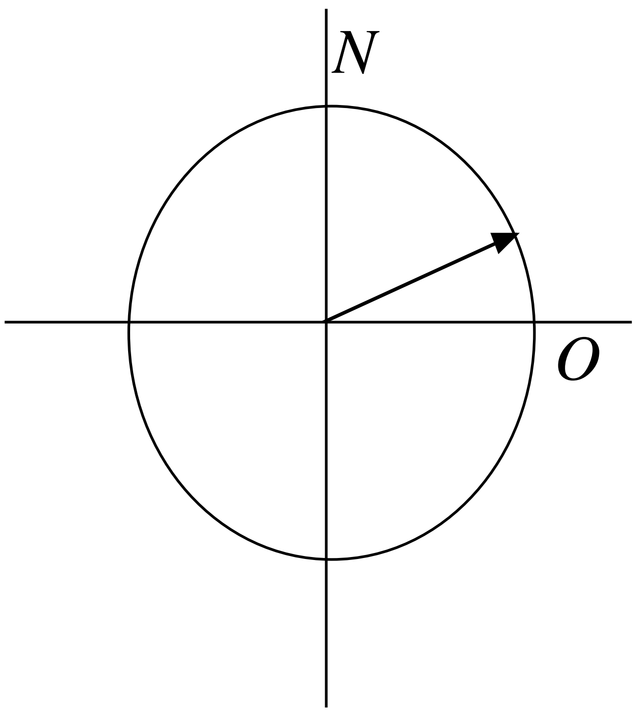
$k, i \perp O\,, j$

Orthogonal Plane



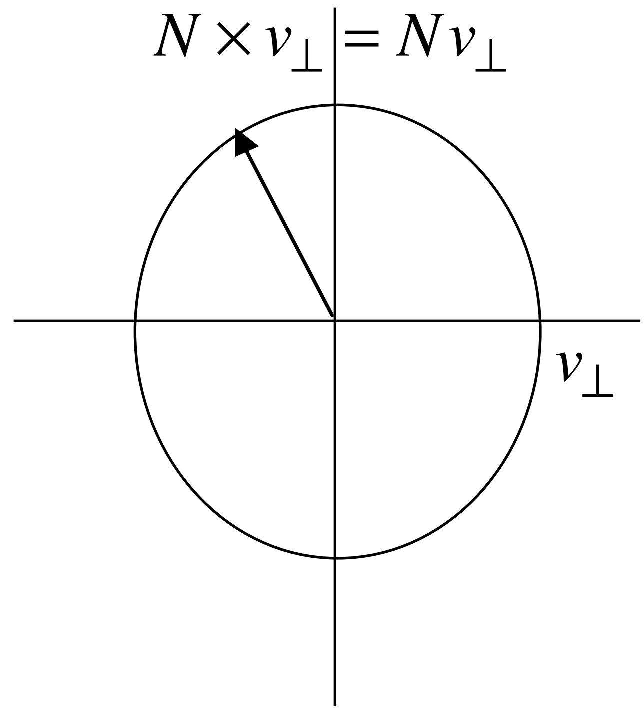
Orthogonal Plane *Complex Plane* * k

Pairs of Complementary Orthogonal Planes



Complex Plane

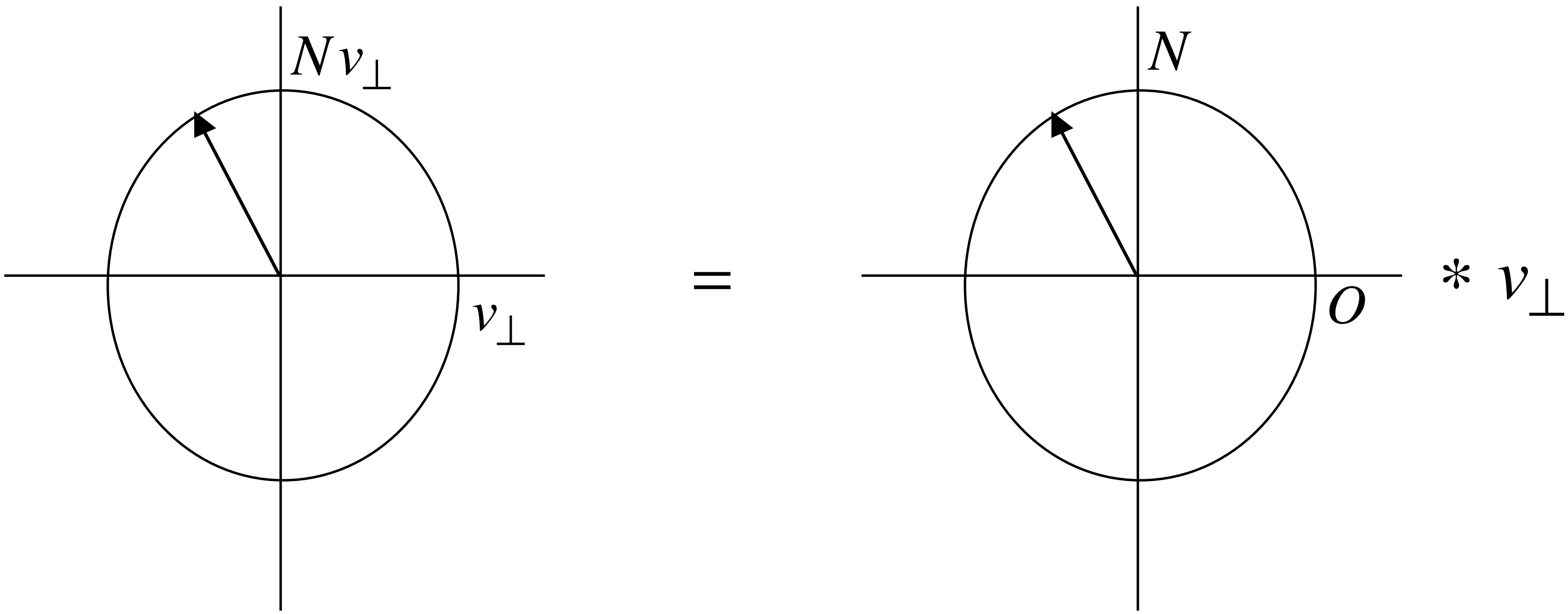
$N^2 = -O\,,\;\; O^2 = O$



Orthogonal Plane

$v_\perp, N \times v_\perp \perp O, N$

Orthogonal Plane



Orthogonal Plane *Complex Plane* $* \; v_{\perp}$

The Geometry of Quaternion Multiplication

Quaternion Multiplication and Isometries

Norm of Product

- $\| pq \| = \| p \| \| q \|$

<u>**Quaternion Multiplication and Isometries**</u>

Norm of Product

- $\| pq \| = \| p \| \| q \|$

Multiplication by Unit Quaternions

- $p \mapsto pq$

- $\| q \| = 1 \Rightarrow$ multiplication by q (on left or right) is a linear isometry in R^4

<u>**Quaternion Multiplication and Isometries**</u>

Norm of Product

- $\| pq \| = \| p \| \| q \|$

Multiplication by Unit Quaternions

- $p \mapsto pq$

- $\| q \| = 1 \Rightarrow$ multiplication by q (on left or right) is a linear isometry in R^4

  $\Rightarrow$ multiplication by q (on left or right) is rotation in R^4

Properties of Vector Multiplication

Vector Multiplication

- $v\,w = (-v \cdot w)O + v \times w$

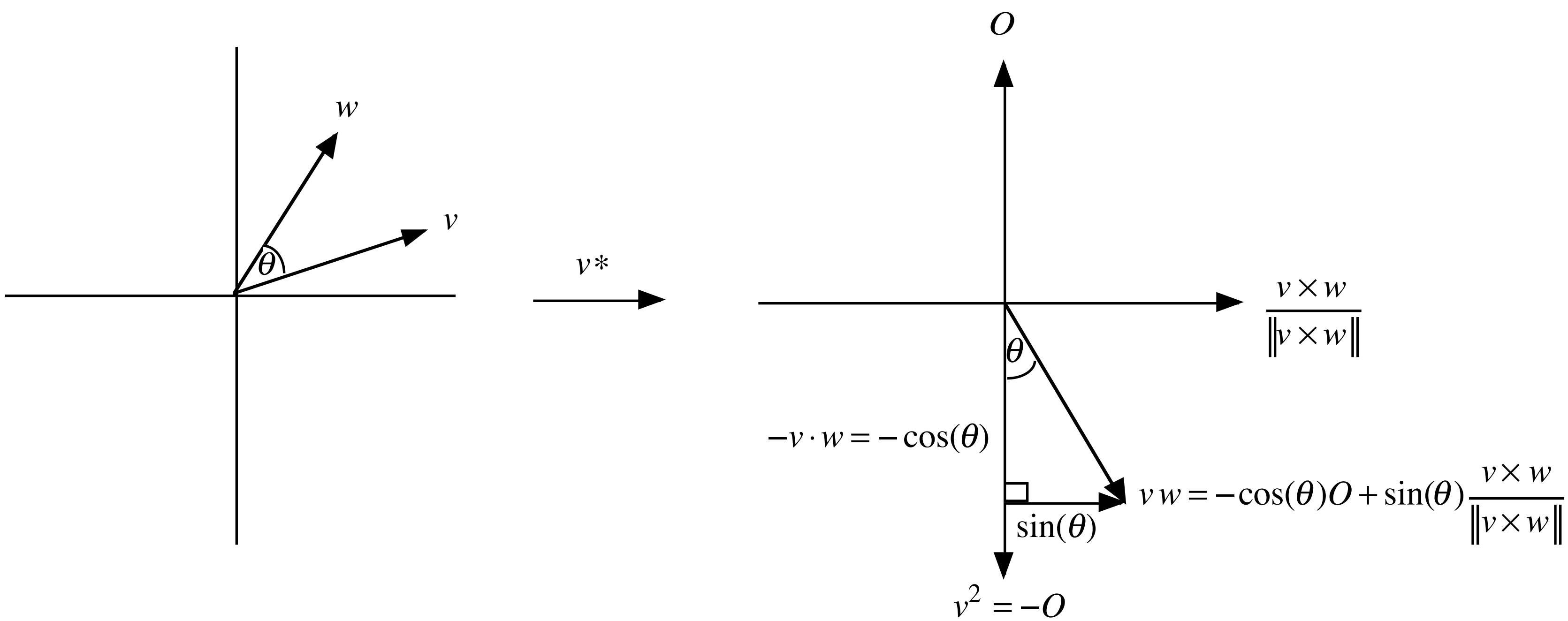
Consequences

- $N \perp v \Rightarrow Nv = N \times v$
- $\| N \| = 1 \Rightarrow N^2 = -O$
- O, N plane is isomorphic to the complex plane

  -- $N^2 = -O\,, \quad O^2 = O$

  -- $NO = ON = N$

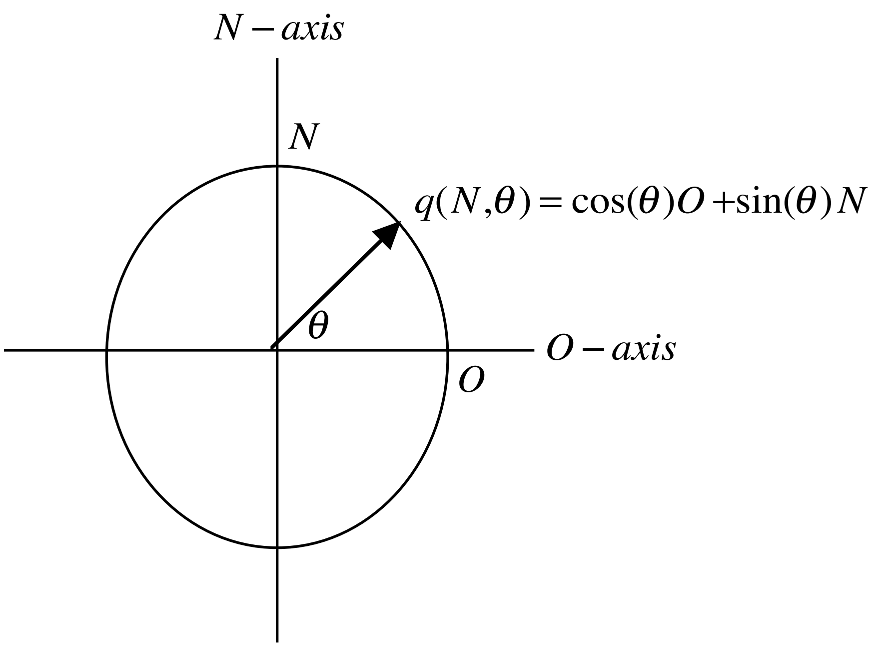
<u>Vector Multiplication Introduces Mass via Rotation</u>



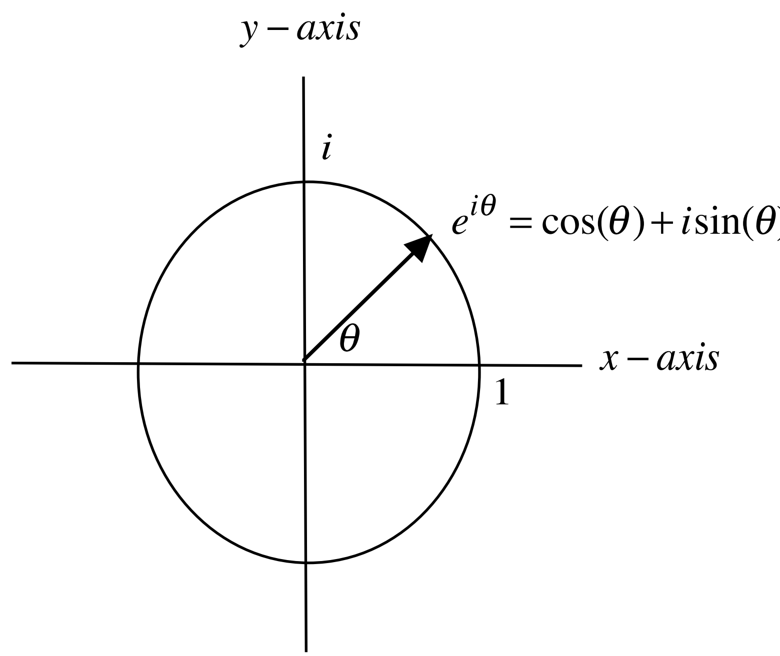
Plane of v,w *Orthogonal Plane of $v\times w, O$*

Multiplication by v represents a rotation in 4-dimensions

Planes Isomorphic to the Complex Plane



O, N Plane



Complex Plane

$N^2 = -O\ ,\ O^2 = O$ $\qquad\qquad$ $i^2 = -1\ ,\ 1^2 = 1$

$q(N,\theta) = \cos(\theta)O + sin(\theta)N \;\leftrightarrow\; Rotation \;\leftrightarrow\; e^{i\theta} = \cos(\theta) + i\,sin(\theta)$

$\{e^{N\theta}\}$ $\qquad\qquad$ *by* θ

Conjugation

Definition

- $q = aO + bi + cj + dk$
- $q^* = aO - bi - cj - dk$

Properties

- $(pq)^* = q^* p^*$
- $q q^* = \| q \|^2 \, O \qquad \left(\Rightarrow \|pq\| = \|p\| \, \|q\| \right)$

Inverses and Inversion

- $q^{-1} = \dfrac{q^*}{q q^*} = \dfrac{q^*}{\| q \|^2}$ (inverses)
- $v^{-1} = -\dfrac{v}{\| v \|^2}$ (inversion)

Inversion in the Sphere

Definition

- q = center of unit sphere
- $inv_q(p) =$ point along line from q to p at distance $d = 1/\|p-q\|$ from q

Formula

- $inv_q(p) = q + \dfrac{p-q}{\|p-q\|^2} = q - (p-q)^{-1}$
- $\|inv_q(p) - q\| = \dfrac{\|p-q\|}{\|p-q\|^2} = \dfrac{1}{\|p-q\|}$

Properties

- inv_q turns unit sphere centered at q inside out
- $inv_q \circ inv_q = identity$
- inv_q maps spheres and planes to spheres and planes

Conjugates of Complex Numbers

Complex Number

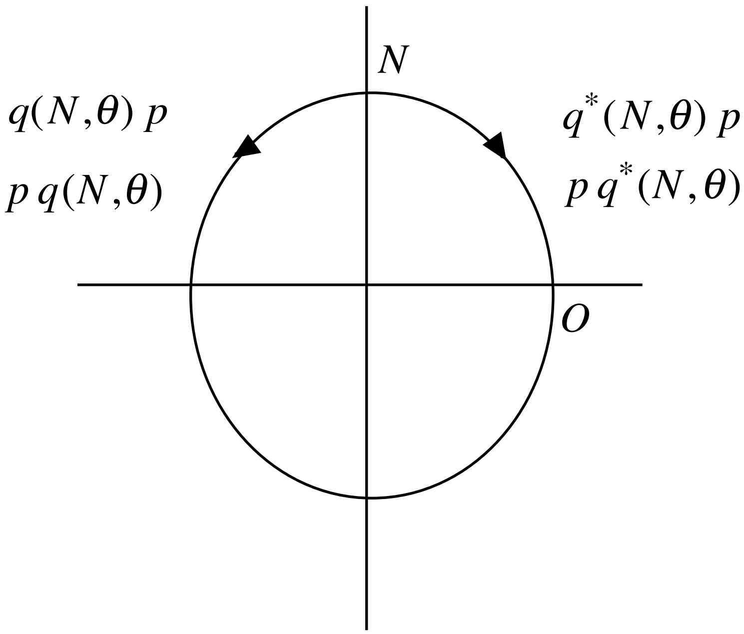
- $q(N,\theta) = \cos(\theta)O + \sin(\theta)N$

Complex Conjugate

- $q^*(N,\theta) = \cos(\theta)O - \sin(\theta)N$

- $q^*(N,\theta) = q(N,-\theta) = q(N,\theta)^{-1}$

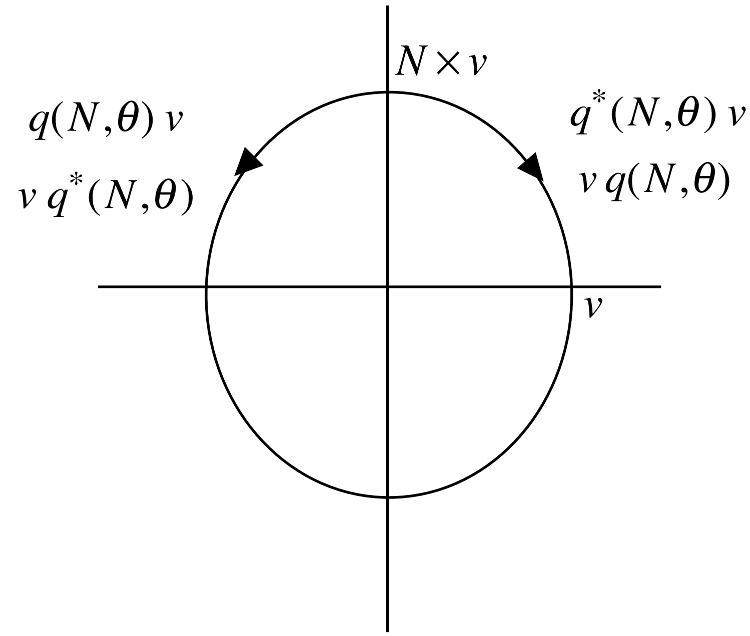
Rotation in Complementary Planes -- Double Isoclinic Rotations



Plane of ***O****, N*

Rotation by the Angle θ

$q(N,\theta),\ q^*(N,\theta)$ *Cancel*



Plane Perpendicular to O, N

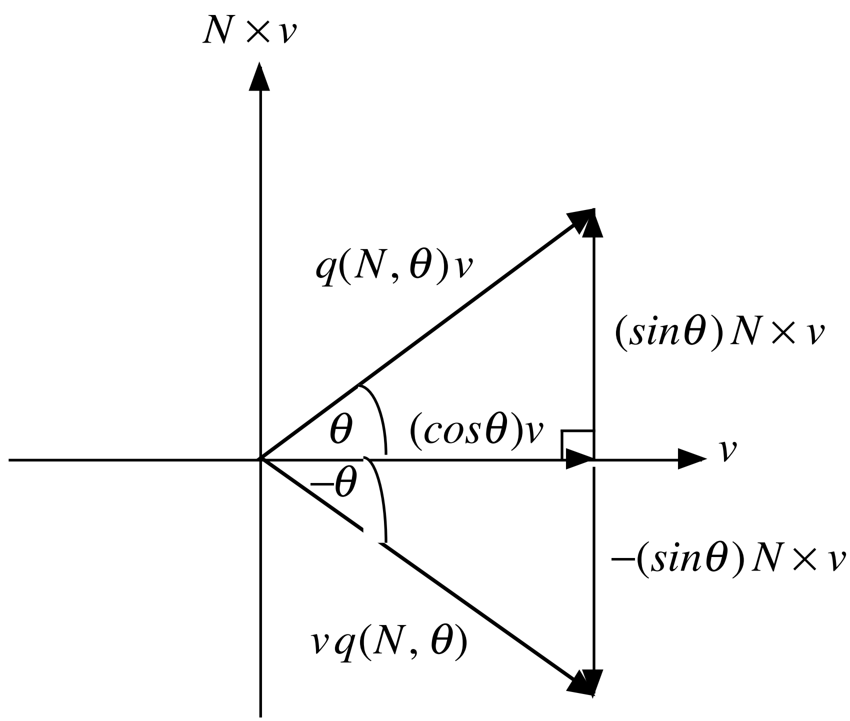
Rotation by the Angle θ

$q(N,\theta),\ q^*(N,\theta)$ *Reinforce*

Rotation in Plane Perpendicular to *O, N*

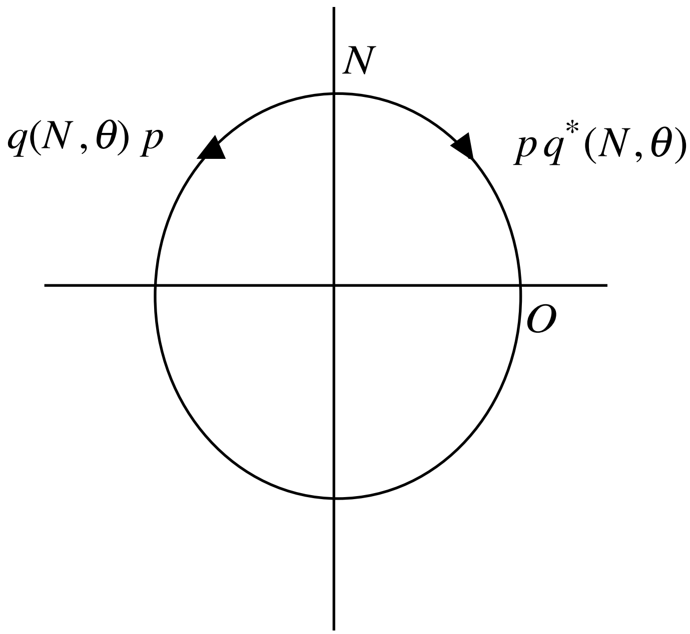
$$i.\quad q(N,\theta)\,v = \big(\cos(\theta)O + \sin(\theta)N\big)\,v = \cos(\theta)v + \sin(\theta)N\times v$$

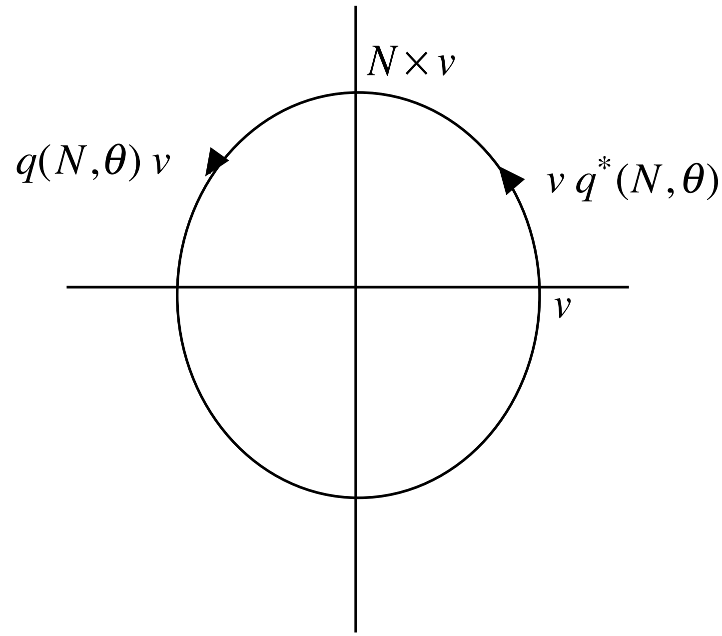
$$ii.\quad v\,q(N,\theta) = v\big(\cos(\theta)O + \sin(\theta)N\big) = \cos(\theta)v + \sin(\theta)v\times N = \cos(\theta)v - \sin(\theta)N\times v$$



Plane $\perp$ *O, N*

Sandwiching with Conjugates in Complementary Planes -- Simple Rotations





Plane of O, N

Sandwiching $q(N,\theta)\, p\, q^*(N,\theta)$

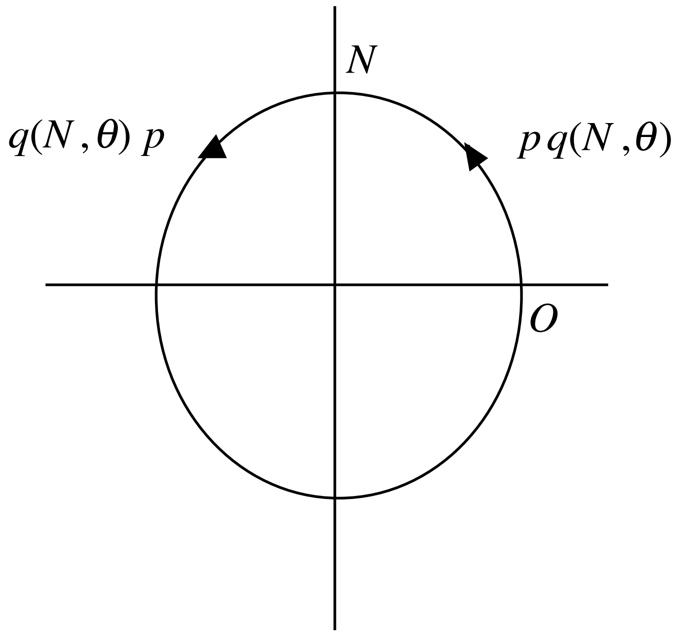
$q(N,\theta),\ q^*(N,\theta)$ *Cancel*

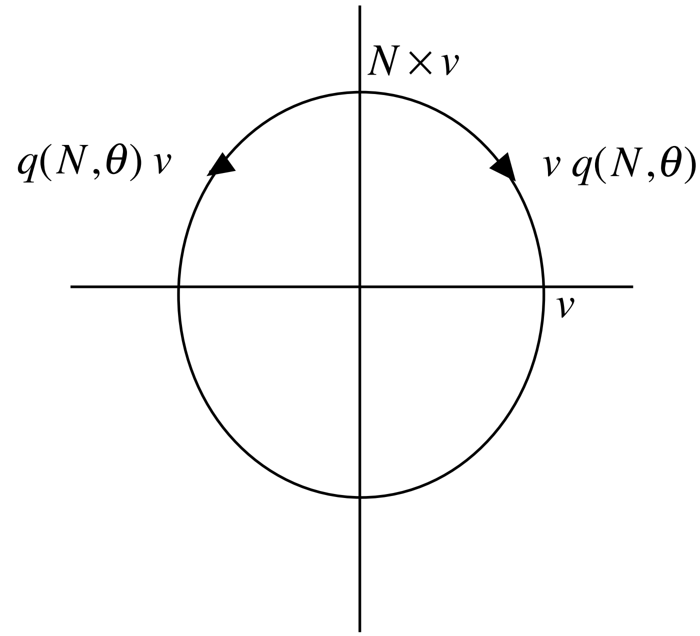
Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\, v\, q^*(N,\theta)$

$q(N,\theta),\ q^*(N,\theta)$ *Reinforce*

Sandwiching in Complementary Planes -- Simple Rotations





Plane of O, N

Sandwiching $q(N,\theta)\,p\,q(N,\theta)$

$q(N,\theta)$ *on Left and Right Reinforce*

Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\,v\,q(N,\theta)$

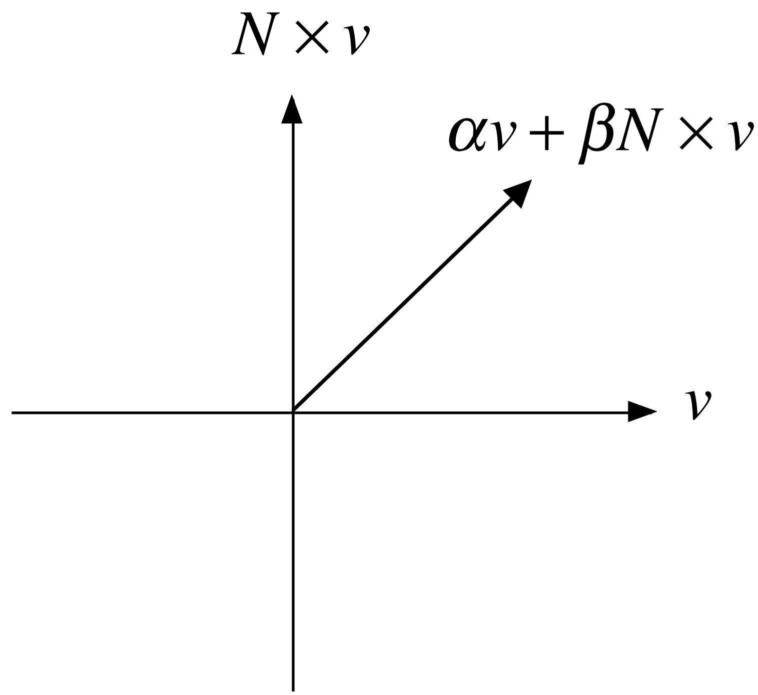
$q(N,\theta)$ *on Left and Right Cancel*

Rotation, Reflection

and

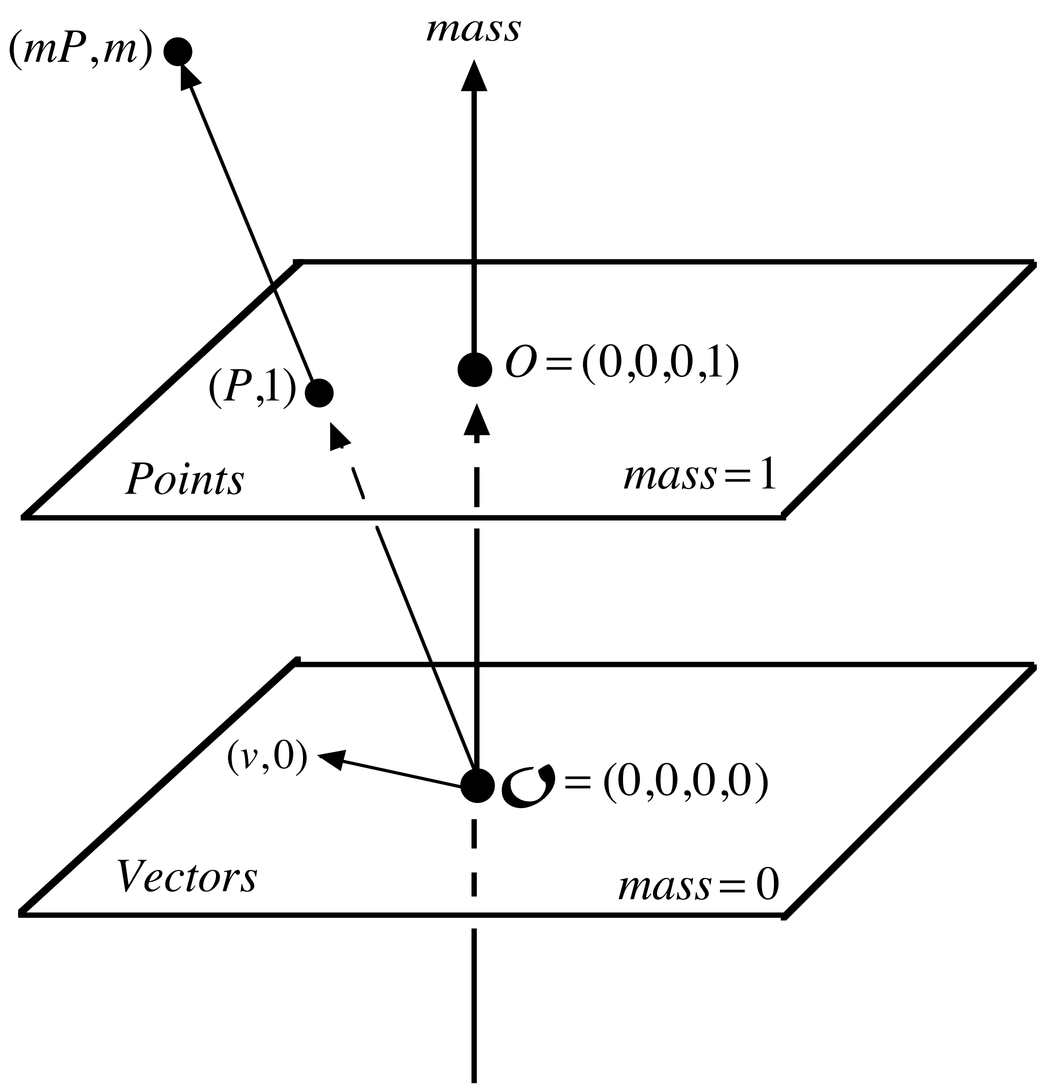
Perspective Projection

3–Dimensional and 4-Dimensional Interpretations of the Plane $\perp$ to *O, N*

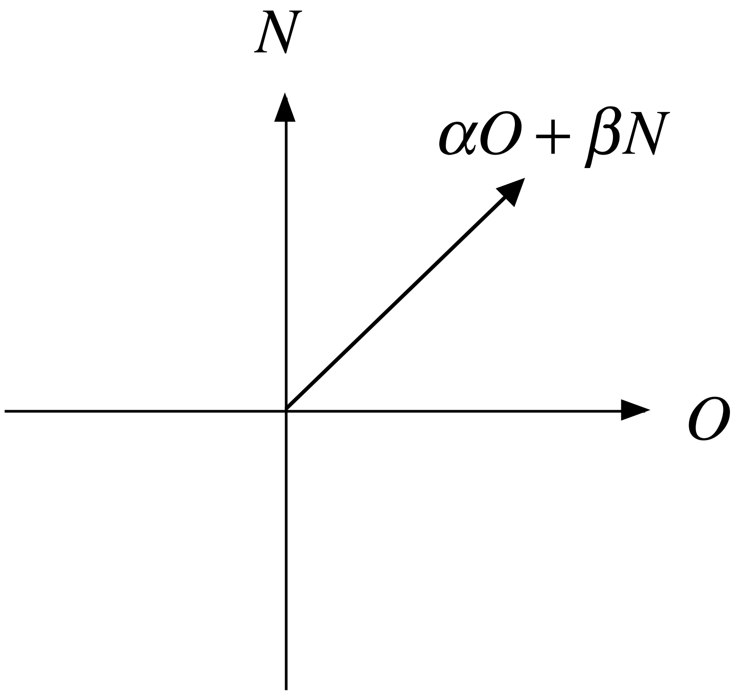


Plane of Vectors $\perp$ O, N in 4-Dimensions = Plane of Vectors $\perp$ N in 3-Dimensions

The 4–Dimensional Vector Space of Quaternions

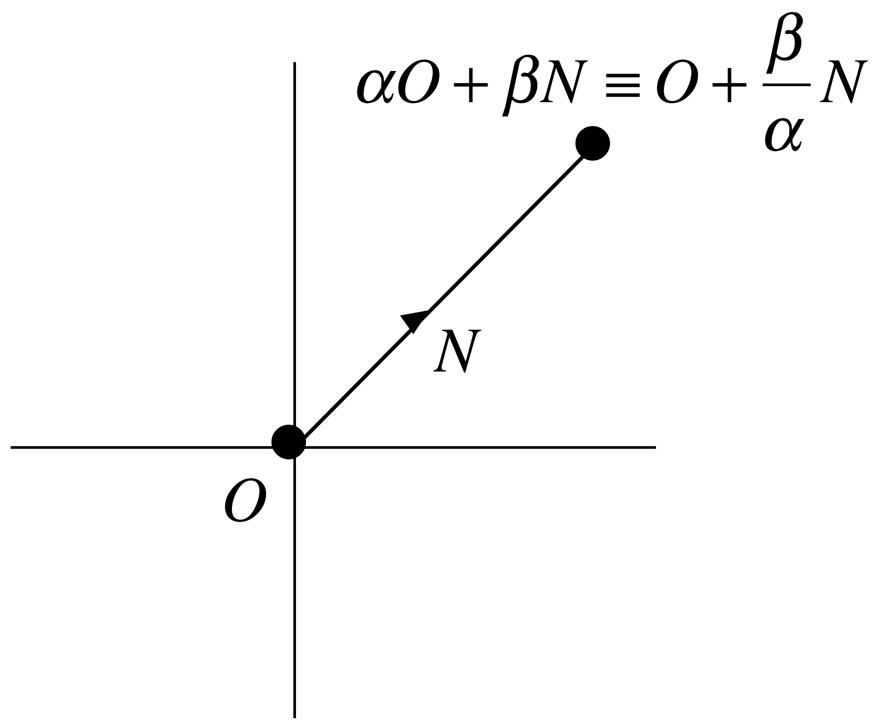


<u>3–Dimensional and 4-Dimensional Interpretations of the Plane of *O*, *N*</u>



Plane of O, N in 4-Dimensions

Plane of Vectors in 4-Dimensions



Line Through O in Direction N in 3-Dimensions

Line of Points in 3-Dimensions

Rotation and Reflection

$p \mapsto q(N,\theta)\, p\, q^*(N,\theta)$

- Plane of O, N = Line through O parallel to N

  -- Identity $\rightarrow$ FIXED AXIS LINE

- Plane $\perp$ O, N = Plane $\perp$ N

  -- Rotation by Angle 2θ -- ROTATION

$p \mapsto q(N,\theta)\, p\, q(N,\theta)$

- Plane $\perp$ O, N = Plane $\perp$ N

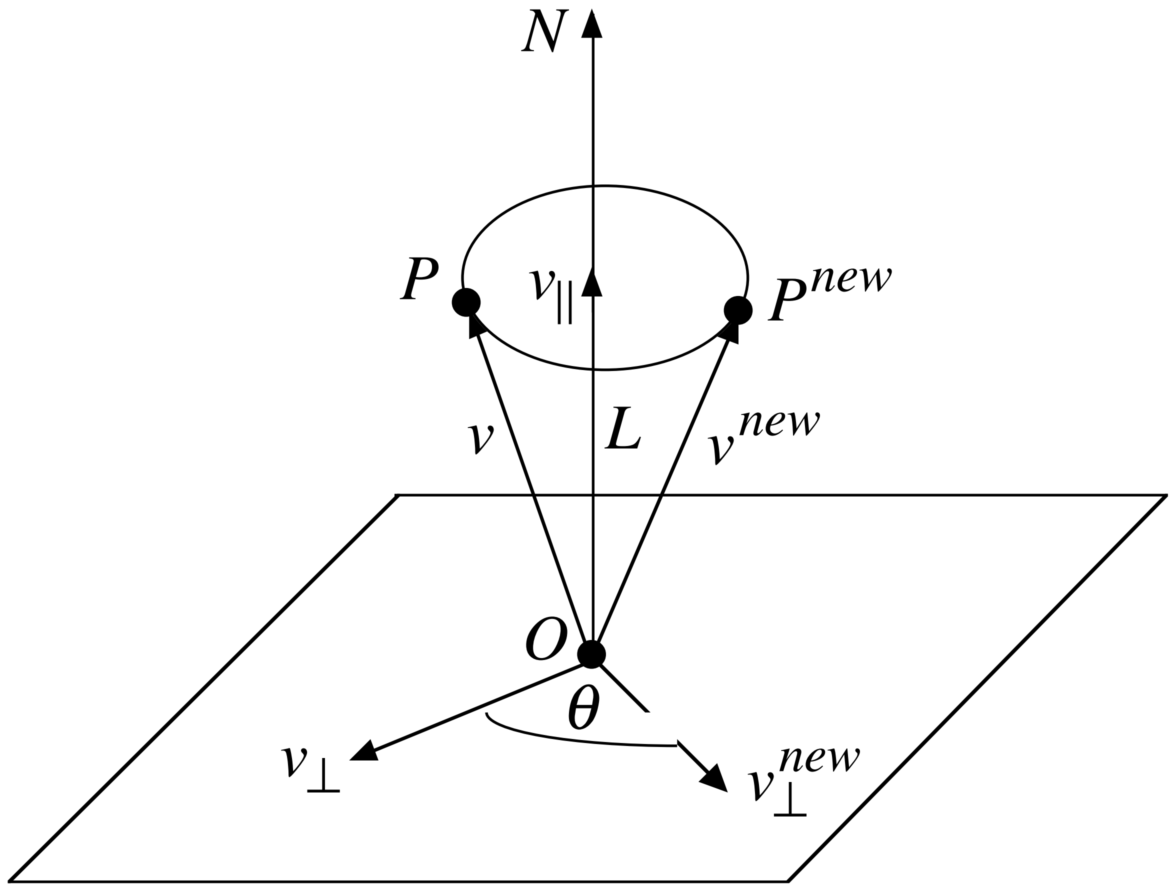
  -- Identity $\rightarrow$ FIXED PLANE

- Plane of O, N = Line through O parallel to N

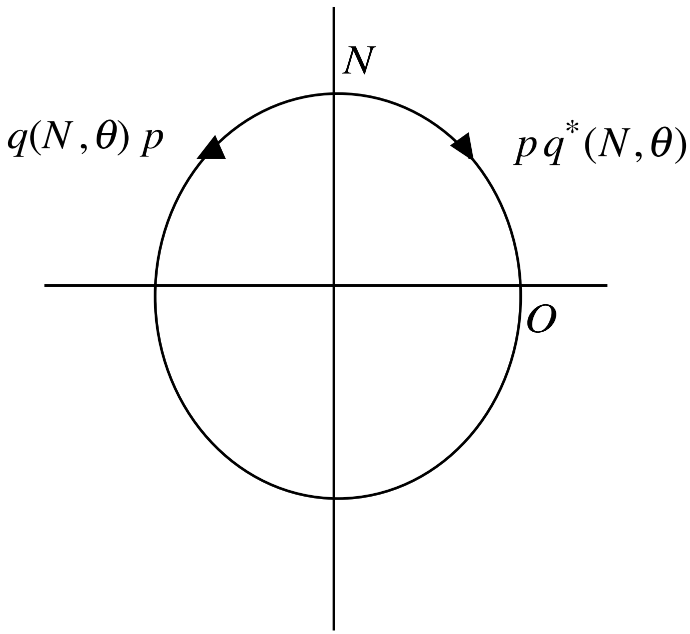
  -- $N \mapsto -N$ -- MIRROR IMAGE

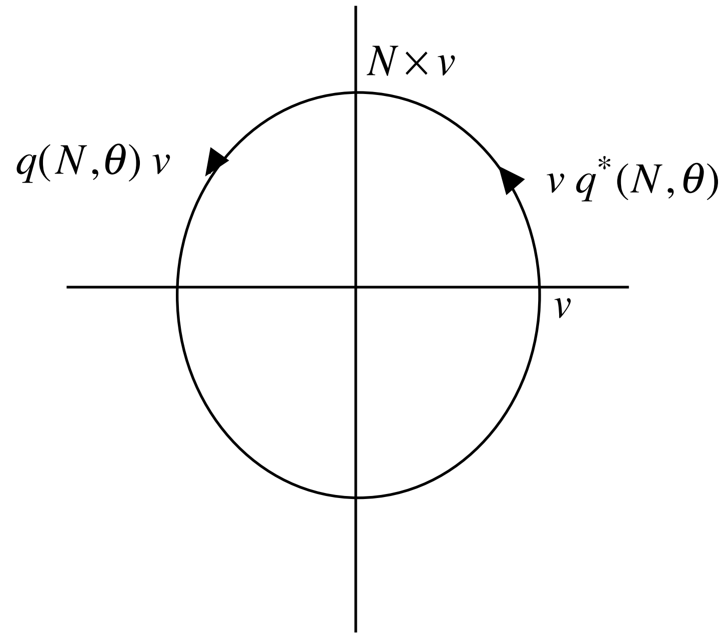
  -- $N \mapsto$ Mass–Point -- PERSPECTIVE PROJECTION

Rotation



Sandwiching with Conjugates in Complementary Planes -- Simple Rotations





Plane of O, N

Sandwiching $q(N,\theta)\, p\, q^{*}(N,\theta)$

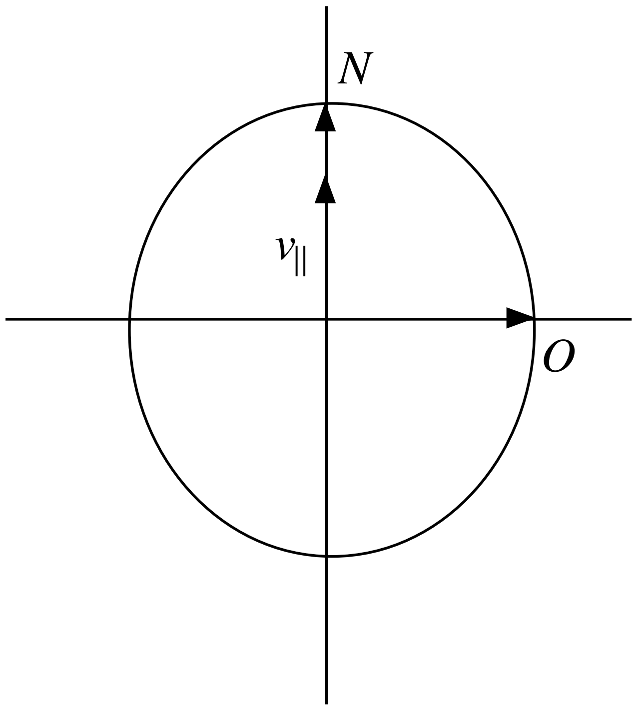
$q(N,\theta)$, $q^{*}(N,\theta)$ *Cancel*

Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\, v\, q^{*}(N,\theta)$

$q(N,\theta)$, $q^{*}(N,\theta)$ *Reinforce*

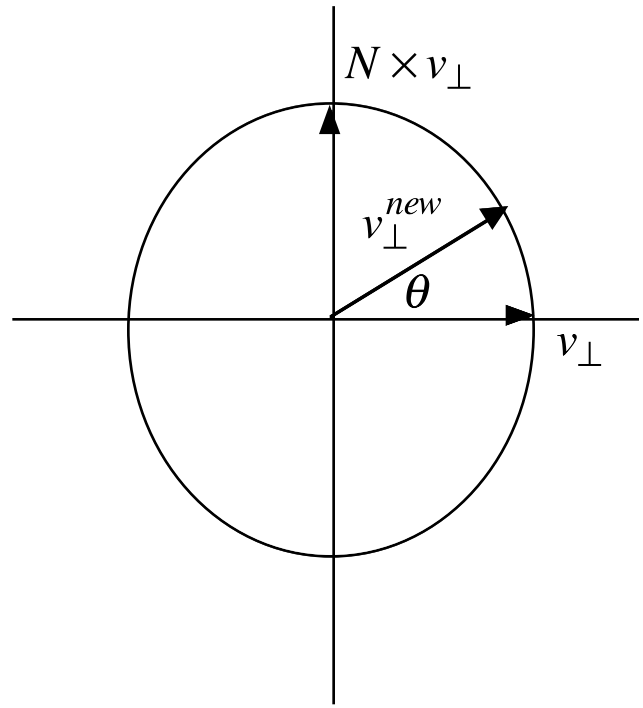
<u>Rotation: Sandwiching in Complementary Planes</u>



<u>Plane of O, N</u>

$$q(N,\theta/2)\; v_{||}\; q^{*}(N,\theta/2) = v_{||}$$

Identity



<u>Plane Perpendicular to O, N</u>

$$q(N,\theta/2)\; v_{\perp}^{new}\; q^{*}(N,\theta/2)$$

Rotation by θ

Theorem 1: Sandwiching Rotates Vectors in 3–Dimensions

Let

- $q(N,\theta/2) = \cos(\theta/2)O + \sin(\theta/2)N$
- v = *vector in* R^3

Then

- $q(N,\theta/2)\, v\, q^*(N,\theta/2)$ *rotates v by the angle* θ *around the axis N*

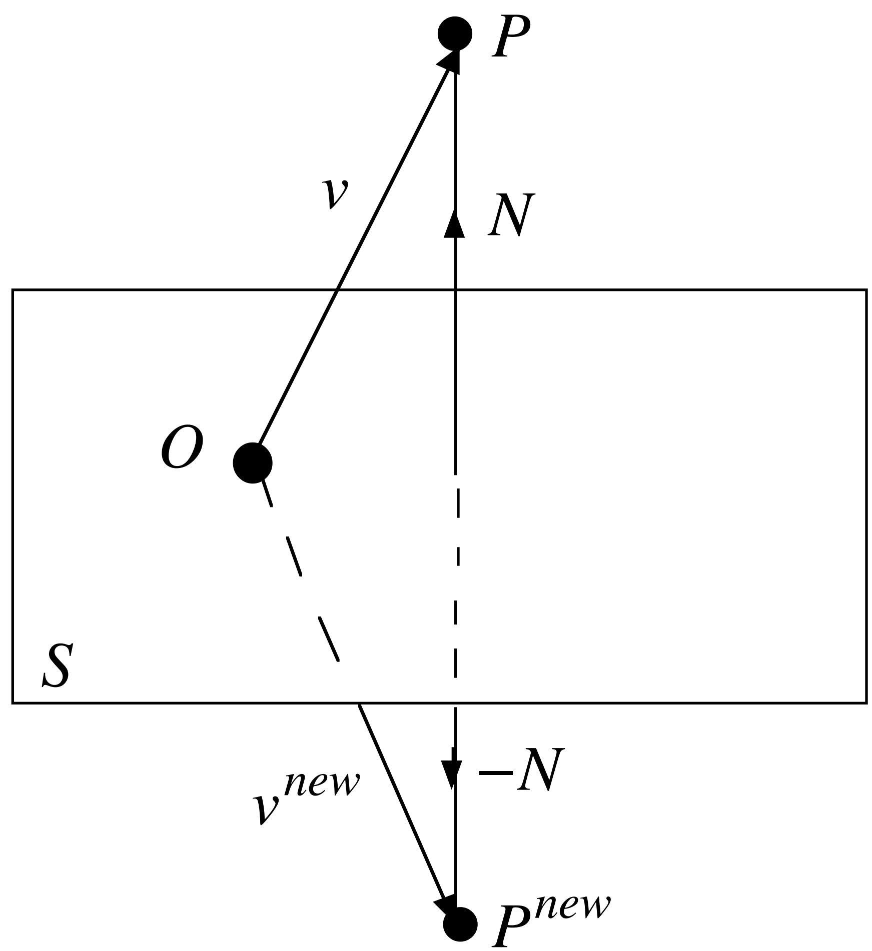
Corollary: Composites of Rotations are Represented by Products of Quaternions

The composite of rotations represented by two quaternions $q(N_1, \theta_1/2)$, $q(N_2, \theta_2/2)$ *is represented by the product quaternion* $q = q(N_2, \theta_2/2)\, q(N_1, \theta_1/2)$.

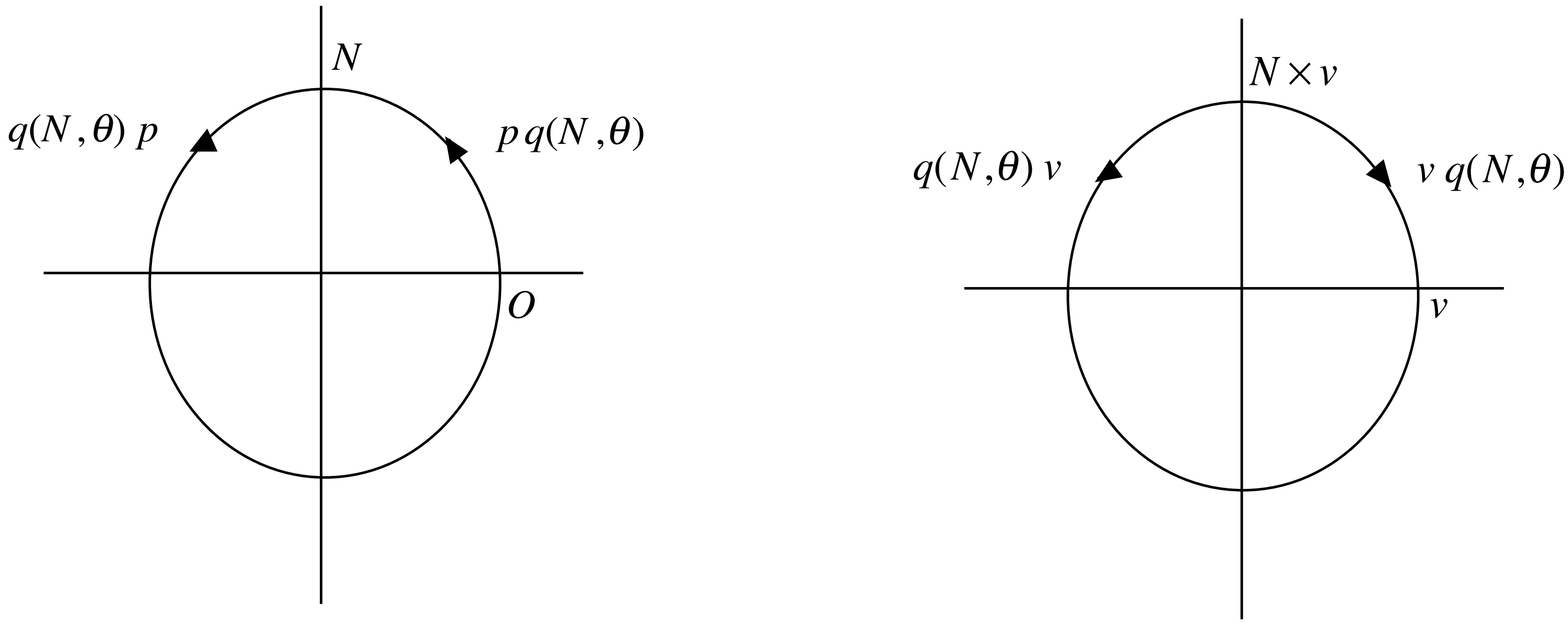
Proof:
$$q\, v\, q^* = q(N_2, \theta_2/2)\, q(N_1, \theta_1/2)\, v \left(q(N_2, \theta_2/2) q(N_1, \theta_1/2)\right)^*$$

$$= q(N_2, \theta_2/2)\, q(N_1, \theta_1/2)\, v\, q^*(N_1, \theta_1/2)\, q^*(N_2, \theta_2/2)$$

Reflection



Sandwiching in Complementary Planes -- Simple Rotations



Plane of O, N

Sandwiching $q(N,\theta)\, p\, q(N,\theta)$

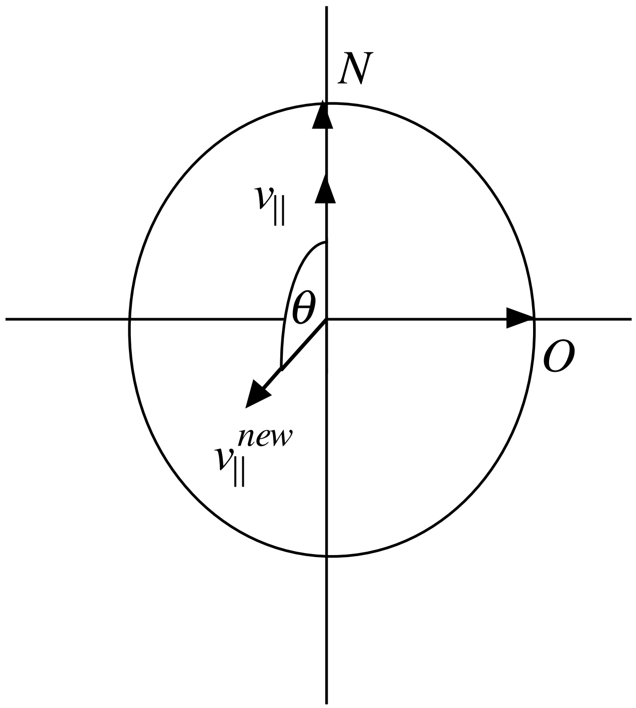
$q(N,\theta)$ *on Left and Right Reinforce*

Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\, v\, q(N,\theta)$

$q(N,\theta)$ *on Left and Right Cancel*

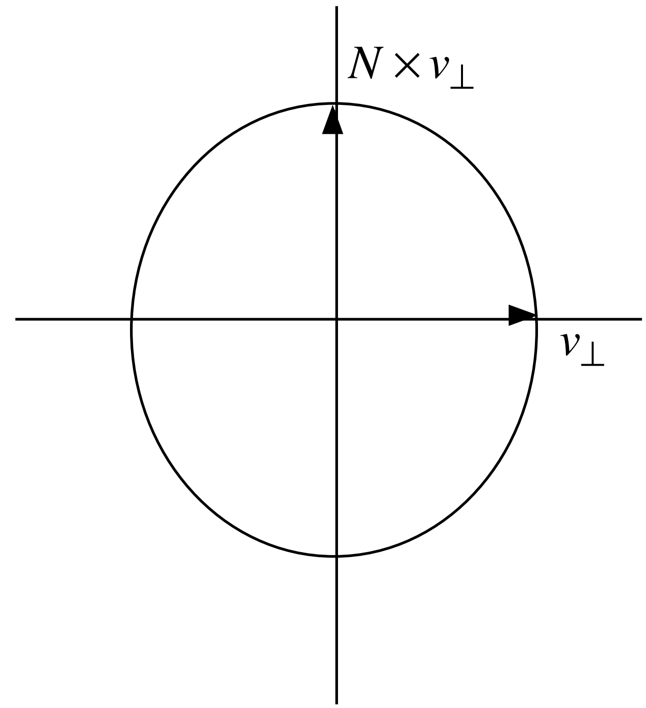
Reflection: Sandwiching in Complementary Planes



Plane of O, N

$$q(N,\theta/2)\ v_{||}\ q(N,\theta/2)$$

Rotation by θ



Plane Perpendicular to O, N

$$q(N,\theta/2)\ v_{\perp}\ q(N,\theta/2) = v_{\perp}$$

Identity

Theorem 2: Sandwiching Reflects Vectors in 3–Dimensions

Let v = *vector in* R^3

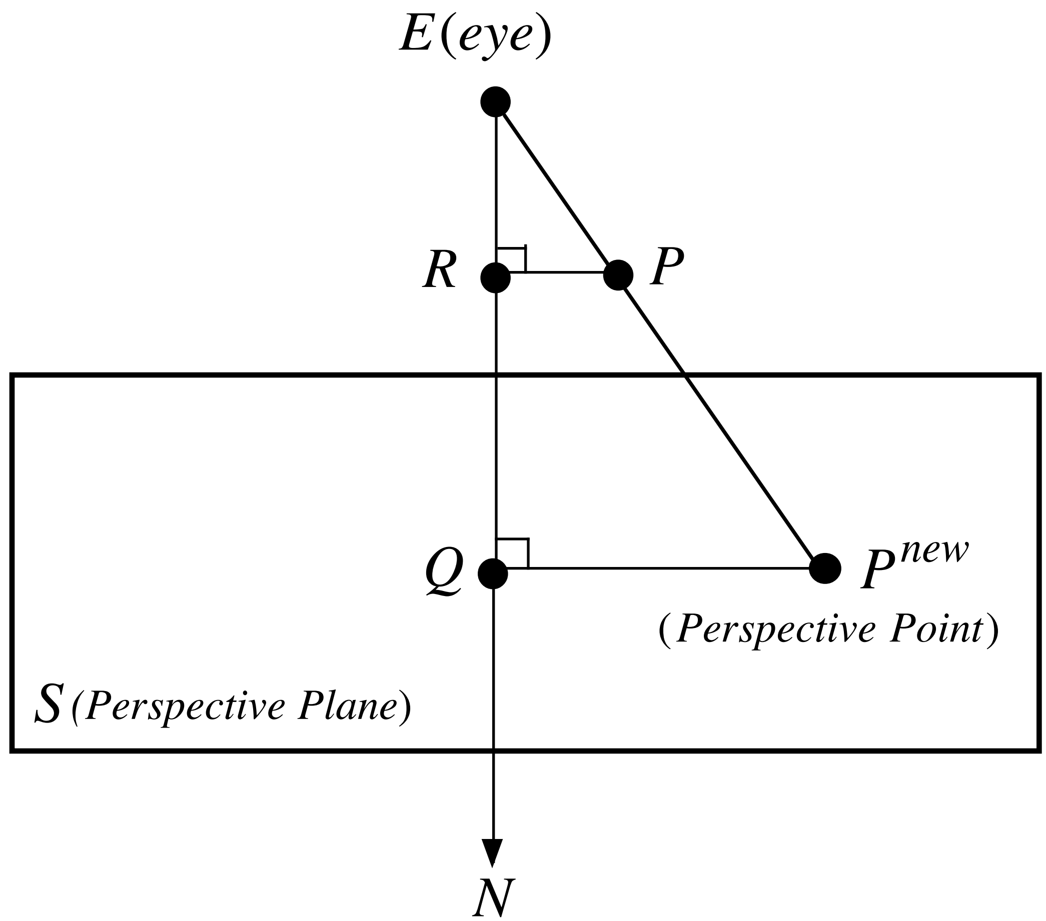
Then $N\,v\,N$ *is the mirror image of w in the plane* $\perp$ N

Proof: Take $\theta = \pi$. Then sandwiching v with:

$$q(N,\pi/2) = \cos(\pi/2)O + \sin(\pi/2)N = N$$

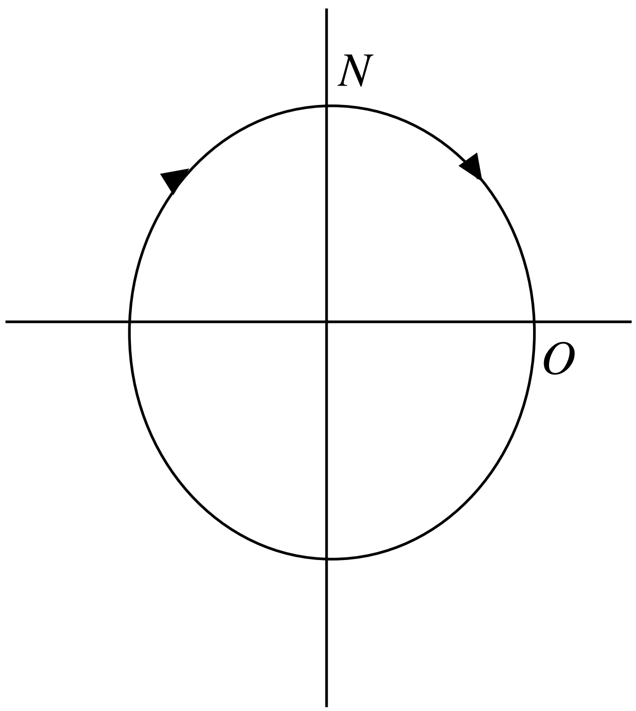
gives the mirror image of v in the plane $\perp$ N.

Perspective Projection

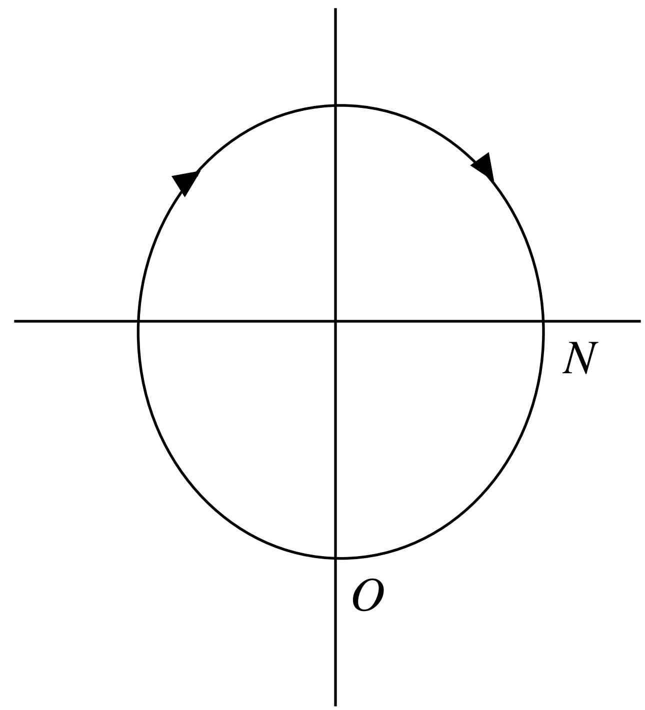


$$\Delta EQP^{new} \approx \Delta ERP$$

Perspective Projection: Sandwiching in the Plane of *O, N*



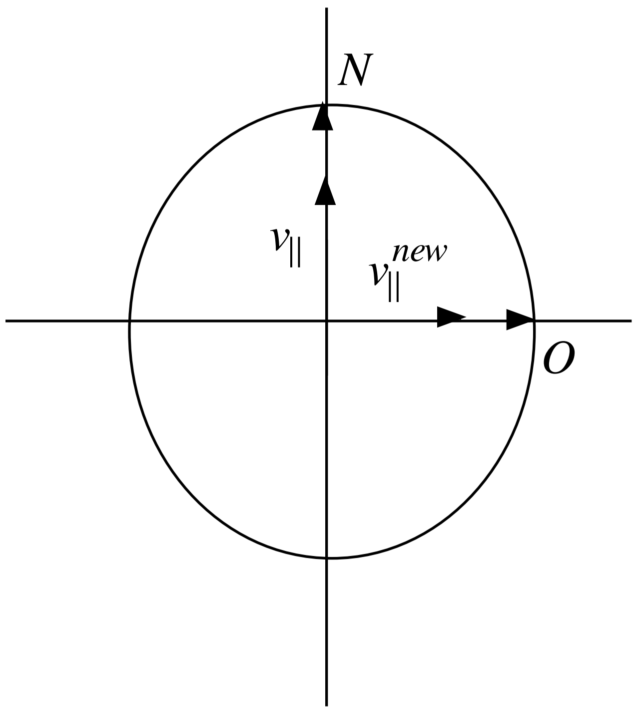
Plane of O, N
Before Sandwiching with $q(N,-\pi/4)$

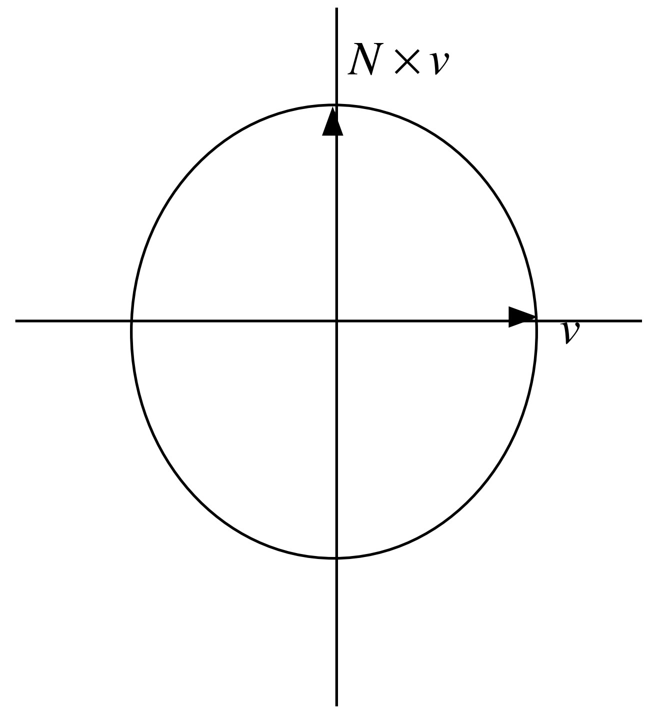


Plane of O, N
After Sandwiching with $q(N,-\pi/4)$

Length along N is mapped to mass at O

Perspective Projection: Sandwiching in Complementary Planes





Plane of O, N

$$q(N,-\pi/4)\ dN\ q(N,-\pi/4) = d\,O$$

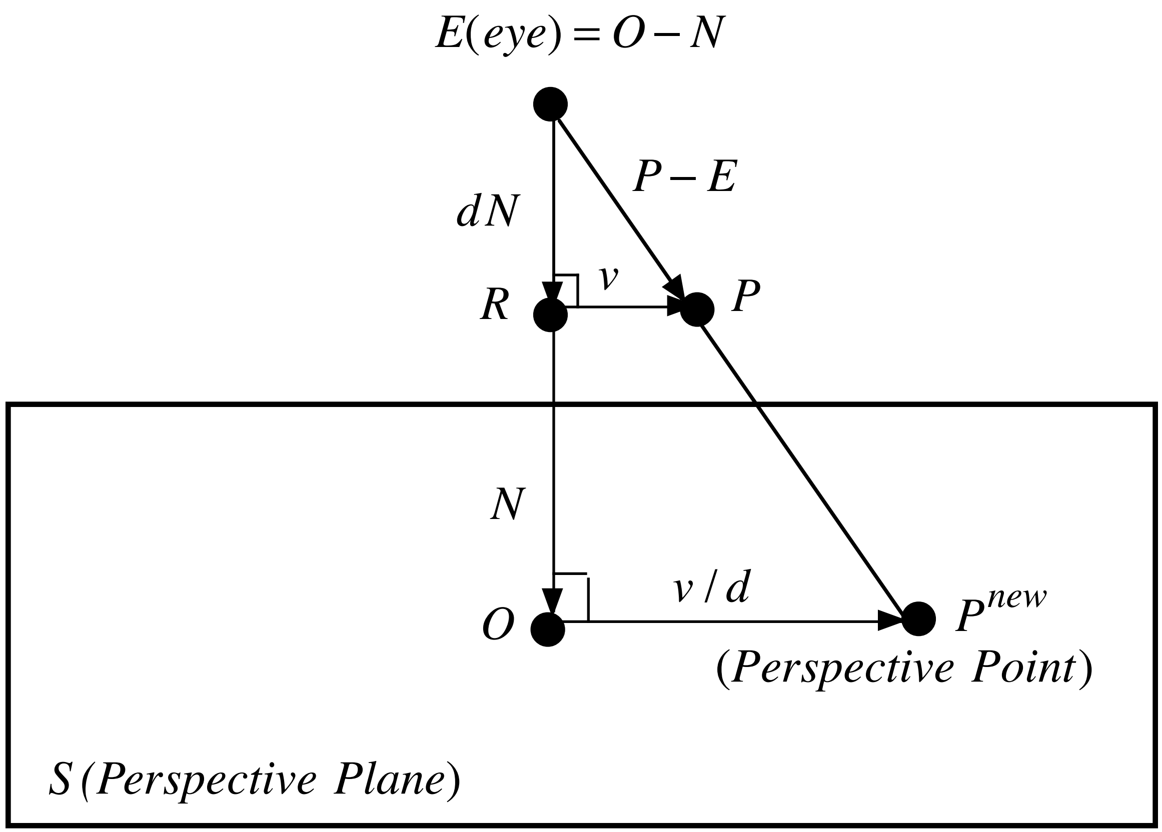
Rotation by $\pi/2$

Plane Perpendicular to O, N

$$q(N,-\pi/4)\ v\, q(N,-\pi/4) = v$$

Identity

Perspective Projection



$$P - E = dN + v \rightarrow dO + v \rightarrow O + v/d$$

$$\Delta EOP^{new} \approx \Delta ERP$$

Theorem 3: Sandwiching Vectors to the Eye with $q(N,-\pi/4)$ Gives Perspective

Let

- *S = plane through the origin O perpendicular to the unit normal N*
- $E = O - N =$ *eye point*
- *P = point in* R^3

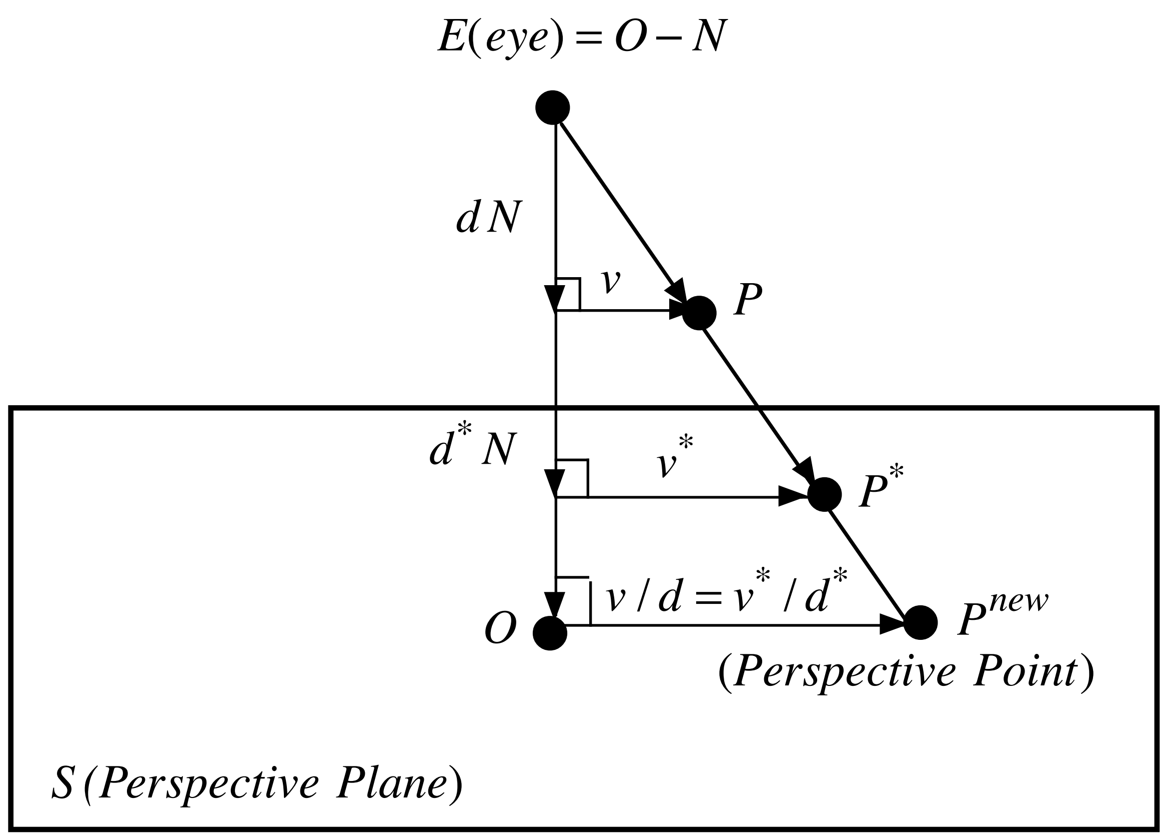
Then

- $q(N,-\pi/4)\,(P-E)\,q(N,-\pi/4)$ *is a mass-point, where:*

  -- *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*

  -- *the mass is equal to the distance d of the point P from the plane through the eye point E perpendicular to the unit normal N.*

Hidden Surfaces



$$d < d^* \Rightarrow P \text{ obscures } P^*$$

Converts Distance Along N to Mass at O

Summary: Sandwiching with $q(N,-\pi/4)$

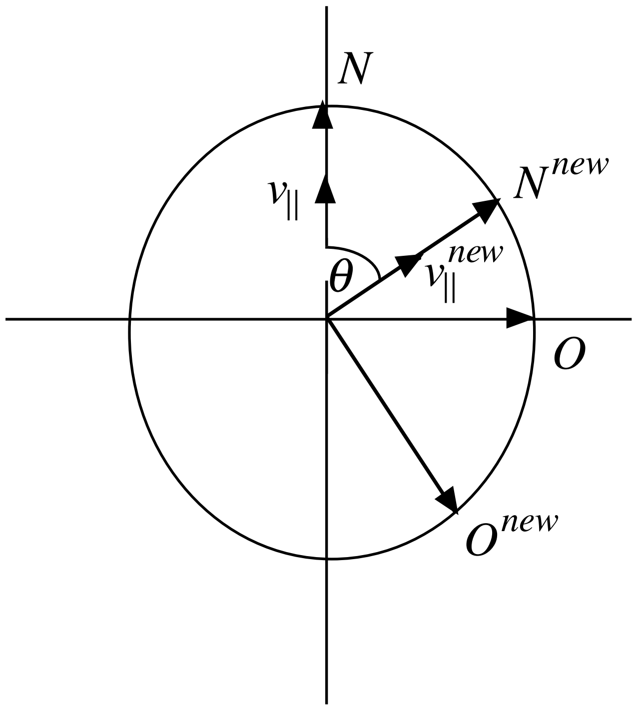
Maps the Vector N to the Point O

- $q(N,-\pi/4)\ N\ q(N,-\pi/4) = O$
- Projects a Vector to a Point
- Projects Points into a Plane

Converts Distance Along N to Mass at O

- $q(N,-\pi/4)\ dN\ q(N,-\pi/4) = d\,O$
- No Information is Lost
- Hidden Surfaces

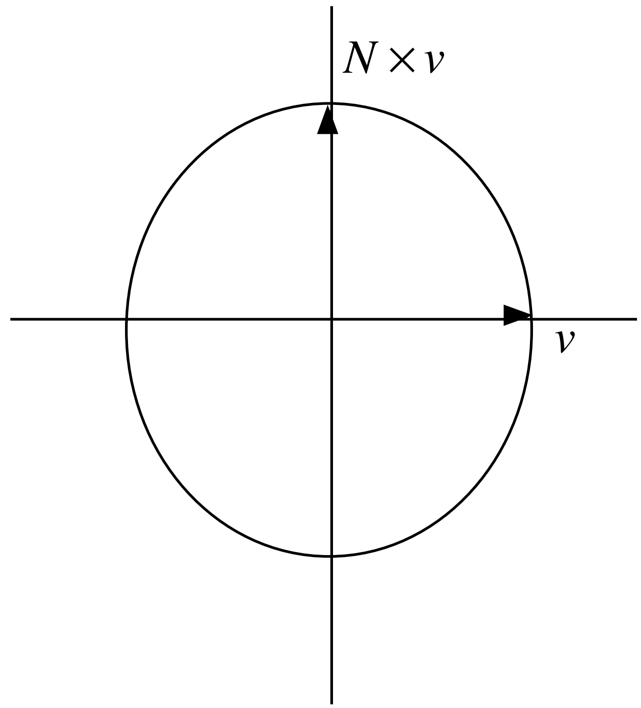
Perspective Projection: Sandwiching in Complementary Planes



Plane of O, N

$$q(N,-\theta/2)\ N\ q(N,-\theta/2) = \sin(\theta)O + \cos(\theta)N$$

Rotation by $-\theta$

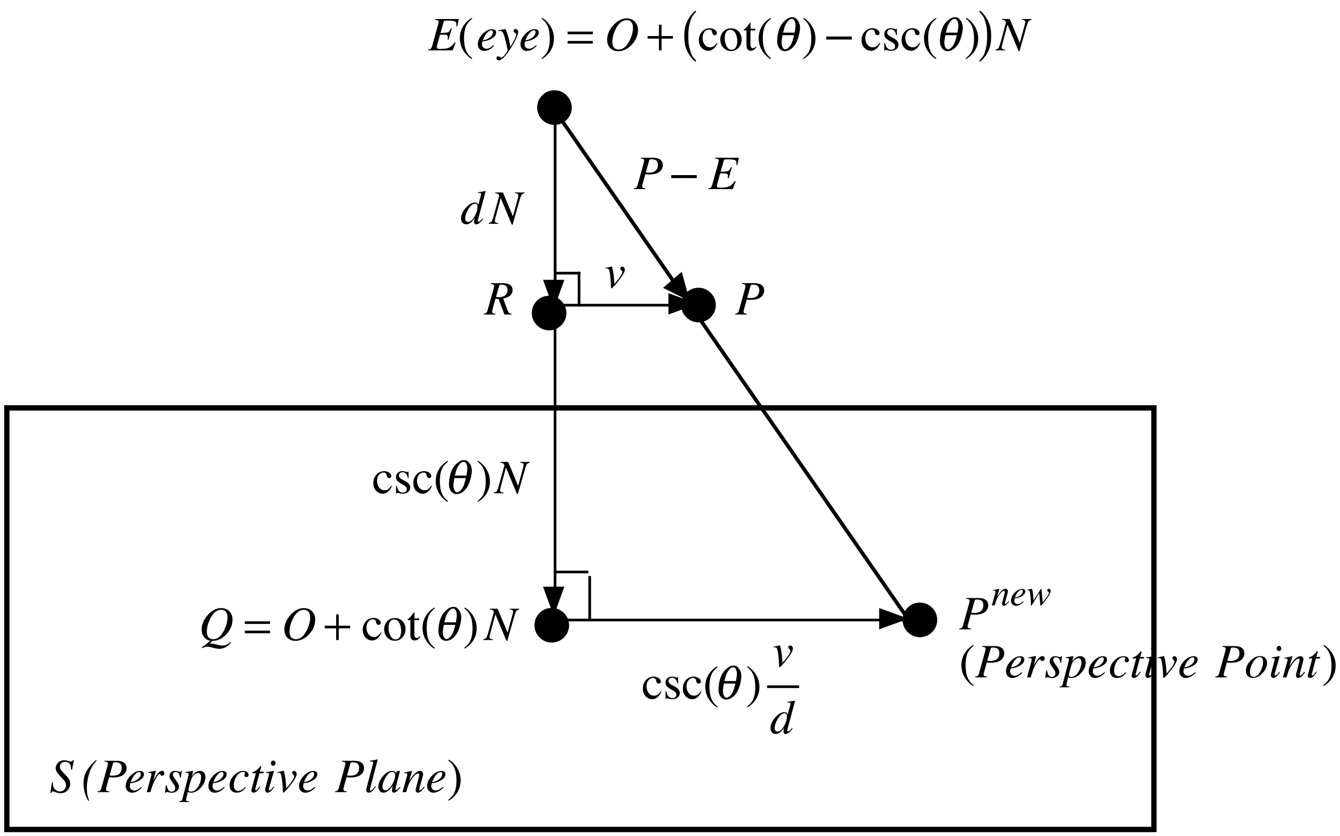


Plane Perpendicular to O, N

$$q(N,\theta/2)\ v\ q(N,\theta/2) = v$$

Identity

Perspective Projection



$$P - E = dN + v \longrightarrow d\sin(\theta)O + d\cos(\theta)N + v \equiv O + \cot(\theta)N + \csc(\theta)\frac{v}{d}$$

$$\Delta EQP^{new} \approx \Delta ERP$$

Theorem 4: Sandwiching Vectors to the Eye with $q(N,-\theta/2)$ **Gives Perspective**

Let

- S = *plane through the point* $O+\cot(\theta)N \equiv q(N,-\theta/2)\; N\; q(N,-\theta/2)$ *perpendicular to the unit normal N*
- $E = O + \left(\cot(\theta) - \csc(\theta)\right)N =$ *eye point*
- P = *point in* R^3

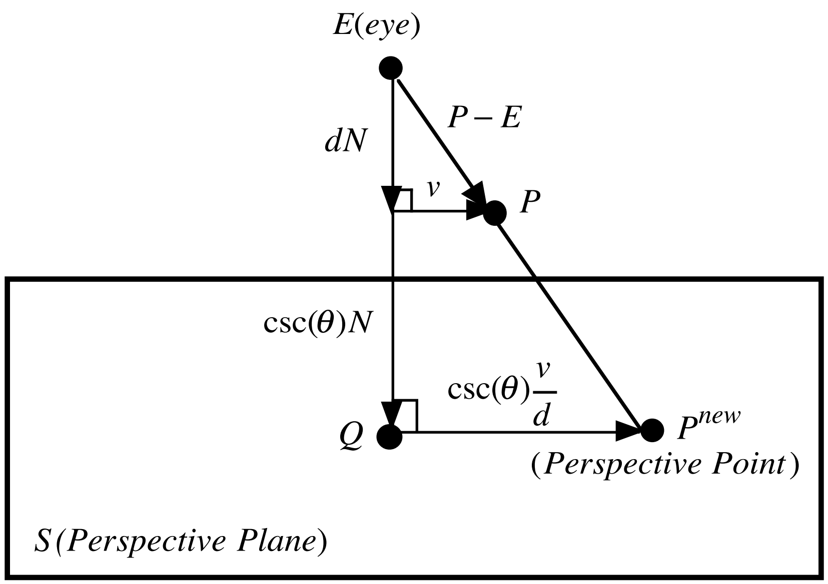
Then

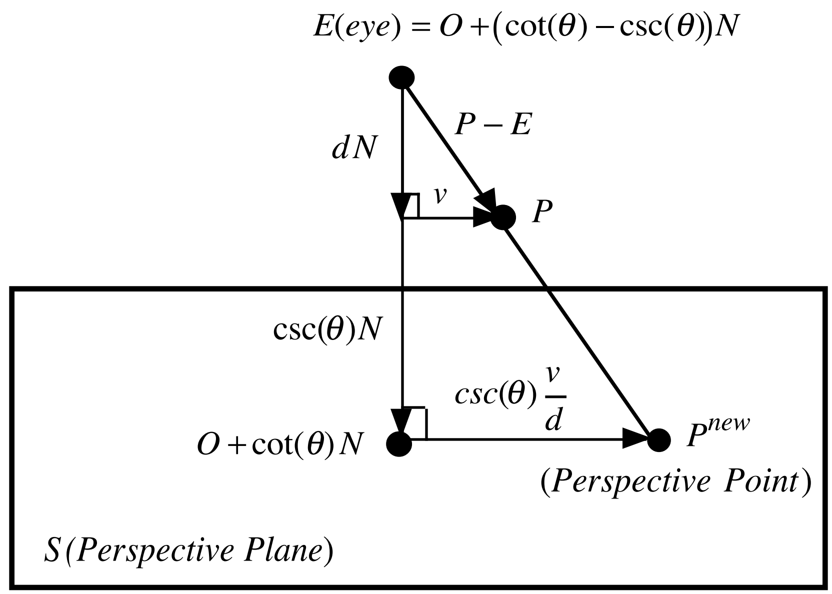
- $q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2)$ *is a mass-point, where:*

  -- *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*

  -- *the mass is equal* $\sin(\theta)$ *times the distance d of the point P from the plane through the eye point E perpendicular to the unit normal* N.

Translation and Perspective Commute





Theorem 5: Sandwiching Vectors to the Eye with $q(N,-\theta/2)$ **Gives Perspective**

Let

- $E =$ *eye point*
- *S = plane at a distance* $\csc(\theta)$ *from E perpendicular to the unit normal N*
- *P = point in* R^3

Then

- $q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2)$ *is a mass-point, where:*

  -- *the point is located at the perspective projection of the point P from the eye point E onto the plane S translated to the canonical plane;*

  -- *the mass is equal* $\sin(\theta)$ *times the distance d of the point P from the plane through the eye point E perpendicular to the unit normal N.*

Proof: Translation and Perspective Commute.

Conclusions

Rotations, Reflections, and Perspective Projections in 3–Dimensions can all be Modeled by Simple Rotations in 4–Dimensions

Simple Rotations in 4-Dimensions can be Modeled by Sandwiching either

i. Between a Unit Quaternion and its Conjugate (Rotation)

ii. Between Two Copies of the Same Unit Quaternion (Reflection and Perspective Projection)

Formulas

Rotation

- $v \mapsto q(N,\theta)\, v\, q^*(N,\theta)$

Reflection

- $v \mapsto N\, v\, N$

Perspective Projection

- $P \mapsto q(N,\theta)(P-E)\, q^*(N,\theta)$

Inversion

- $P \mapsto Q-(P-Q)^{-1}$

References

1. Goldman, R. (2010), *Rethinking Quaternions: Theory and Computation,* Synthesis Lectures on Computer Graphics and Animation, ed. Brian A. Barsky, No. 13. San Rafael: Morgan & Claypool Publishers.

2. Goldman, R. (2011), Understanding Quaternions, *Graphical Models*, Vol. 73, pp. 21-49.

3. Goldman, R. (2011), Modeling Perspective Projections in 3-Dimensions by Rotations in 4-Dimensions, submitted for publication.